

Maxwell eigenmodes in product domains

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Abstract. This paper is devoted to Maxwell modes in three-dimensional bounded electromagnetic cavities that have the form of a product of lower dimensional domains in some system of coordinates. The boundary conditions are those of the perfectly conducting or perfectly insulating body. The main case of interest is products in Cartesian variables. Cylindrical and spherical variables are also addressed. We exhibit common structures of polarization type for eigenmodes. In the Cartesian case, the cavity eigenvalues can be obtained as sums of Dirichlet or Neumann eigenvalues of positive Laplace operators and the corresponding eigenvectors have a tensor product form. We compare these descriptions with the spherical wave function Ansatz for a ball and show why the cavity eigenvalue of the ball are also Dirichlet or Neumann eigenvalues of some scalar operators. As application of our general formulas, we find explicit eigenpairs in a cuboid, in a circular cylinder, and in a cylinder with a coaxial circular hole. This latter example exhibit interesting “TEM” eigenmodes that have a one-dimensional vibrating string structure, and contribute to the least energy modes if the cylinder is long enough.

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1 Introduction

A domain Ω of $\mathbb{R}^n$ is called a product domain if for a choice of Cartesian coordinates $x = (y, z)$ in $\mathbb{R}^n$, the domain Ω coincides with the product $\mathcal{Y} \times \mathcal{Z}$ in the sense that

$$x \in \Omega \iff y \in \mathcal{Y} \text{ and } z \in \mathcal{Z}.$$

In the three-dimensional space ($n = 3$), we may assume without restriction that $\mathcal{Y}$ has the dimension 2, and $\mathcal{Z}$, dimension 1, hence is an interval. Such a domain may also be called a cylinder. The main motivation of this work is to exhibit for electromagnetic cavity problems in a cylinder with arbitrary cross section similar properties as those, well know, for acoustic modal problems.

Most of the results we present are not new and have their roots in the pioneering works by Mie (1908) and Debye (1909). Expressions for cavity modes in cylinders can be found in [8] and in balls in [7]. Our aim is to adopt a synthetic presentation that clearly links Laplace or Laplace-like eigenvectors to electromagnetic eigenmodes via TE and TM vector wave functions: The Laplace eigenvectors appear as Debye

potentials. In particular we carefully address the case when the cross section ω of Ω contains holes (modelling for instance metallic wires) and prove the completeness of a system of TE, TM and TEM modes. The TEM eigenmodes that enjoy both features of transverse electric and magnetic polarizations, often contribute the lowest frequencies, and this can be precisely quantified. This case was the first motivation for the present investigation.

The knowledge of Maxwell eigenmodes to an applied mathematics audience has some importance. Our results can be used as benchmarks for numerical methods for the computation of cavity modes. They also allow to localize interior eigenmodes when discretizing transmission problems by methods that fail on interior frequencies.

1.1 The case of acoustics: The Dirichlet Laplacian

The Laplace operator Δ in $\mathbb{R}^n$ is expressed in variables $\mathbf{x} = (x_1, \dots, x_n)$ as $\Delta = \sum_{1 \leq j \leq n} \partial_{x_j}^2$ and it is the sum of the two Laplace operators in variables $\mathbf{y}$ and $\mathbf{z}$

$$\Delta = \Delta_{\mathbf{y}} + \Delta_{\mathbf{z}}.$$

The Sobolev space $H^1(\Omega)$ on the product domain $\Omega = \mathcal{Y} \times \mathcal{Z}$ can be written as

$$H^1(\Omega) = L^2(\mathcal{Y}, H^1(\mathcal{Z})) \cap H^1(\mathcal{Y}, L^2(\mathcal{Z})).$$

Likewise, the closure $H_0^1(\Omega)$ in $H^1(\Omega)$ of smooth functions with compact support in Ω satisfies

$$H_0^1(\Omega) = L^2(\mathcal{Y}, H_0^1(\mathcal{Z})) \cap H_0^1(\mathcal{Y}, L^2(\mathcal{Z})).$$

As a direct consequence we find, for any bounded product domain Ω , the full spectral description of the Dirichlet Laplacian:

Theorem 1.1. *Let $(\lambda_j, v_j)_{j \geq 1}$ and $(\mu_m, w_m)_{m \geq 1}$ be the spectral sequences of $-\Delta_{\mathbf{y}}$ on $H_0^1(\mathcal{Y})$ and of $-\Delta_{\mathbf{z}}$ on $H_0^1(\mathcal{Z})$, respectively. This means that*

$$\lambda_1 < \lambda_2 \leq \dots$$

is the eigenvalue sequence of $-\Delta_{\mathbf{y}}$ and $(v_j)_{j \geq 1}$ is an associated orthonormal basis, and the same for $-\Delta_{\mathbf{z}}$.

Then the set of eigenvalues of $-\Delta$ on $H_0^1(\mathcal{Y} \times \mathcal{Z})$ is

$$\{\lambda_j + \mu_m, \quad j \geq 1, \quad m \geq 1\}$$

and the tensor functions $u_{j,m} := v_j \otimes w_m$, i.e. defined as

$$u_{j,m}(\mathbf{x}) = v_j(\mathbf{y})w_m(\mathbf{z}),$$

are orthonormal associated eigenvectors, and they form a basis of $H_0^1(\Omega)$.

Of course, a similar result holds for Neumann boundary conditions. This holds also for mixed Dirichlet-Neumann problems of the type

$$(\text{Dirichlet on } \partial\mathcal{Y} \times \mathcal{Z}) \quad \text{and} \quad (\text{Neumann on } \mathcal{Y} \times \partial\mathcal{Z})$$

for which the (λ_j, v_j) are still the Dirichlet eigenpairs on $\mathcal{Y}$, but the (μ_m, w_m) have to be taken as the Neumann eigenpairs on $\mathcal{Z}$. Finally the Dirichlet and Neumann conditions can be also be swapped between $\mathcal{Y}$ and $\mathcal{Z}$.

1.2 The case of electromagnetism: The Maxwell system

From now on the space dimension is $n = 3$. Let Ω be a domain in $\mathbb{R}^3$, representing a cavity filled by an homogeneous dielectric medium. We assume that the boundary of Ω represents *perfectly conducting* walls. After normalization, the cavity resonator problem is to find the frequencies $k \in \mathbb{R}$ and the non-zero electromagnetic fields $(\mathbf{E}, \mathbf{H})$ in $L^2(\Omega)^6$ such that

$$\begin{cases} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = 0 & \text{in } \Omega, \\ \operatorname{curl} \mathbf{H} + ik\mathbf{E} = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{E} = 0 \quad \text{and} \quad \operatorname{div} \mathbf{H} = 0 & \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H} \cdot \mathbf{n} = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here, $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. The gauge conditions on the divergence are a consequence of the first two equations if $k \neq 0$. Nevertheless we look for solutions of (1.1) including $k = 0$. The occurrence of $k = 0$ happens if and only if the domain Ω is topologically non-trivial, i.e. if Ω is not simply connected, or if $\partial\Omega$ is not connected, see Propositions 3.14 & 3.18 in reference [1].

Definition 1.2. The triples $(k, \mathbf{E}, \mathbf{H})$ solution of (1.1) with $(\mathbf{E}, \mathbf{H}) \neq 0$ are called Maxwell eigenmodes, k is called eigenfrequency, k^2 eigenvalue, and $\mathbf{E}, \mathbf{H}$ electric and magnetic eigenvectors.

Let Ω be a bounded product domain in $\mathbb{R}^3$. This means that

$$\Omega = \omega \times I, \quad \omega \subset \mathbb{R}^2, \quad I \text{ interval in } \mathbb{R}. \quad (1.2)$$

We denote correspondingly Cartesian coordinates in Ω by

$$x = (x_1, x_2, x_3) = (x_\perp, x_3), \quad x_\perp \in \omega \quad \text{and} \quad x_3 \in I.$$

We assume that ω is a bounded Lipschitz domain. We note that the boundary of Ω is connected. But, if ω is not simply connected, the same holds for Ω .

Notation 1.3. (i) Denote by $\Delta_\perp = \partial_1^2 + \partial_2^2$ the Laplace operator in the variables $x_\perp$.

- (ii) Let $(\lambda_j^{\text{dir}}, v_j^{\text{dir}})_{j \geq 1}$ be the eigenpair sequence of the Dirichlet problem in ω for the operator $-\Delta_\perp$.
- (iii) Let $(\lambda_j^{\text{neu}}, v_j^{\text{neu}})_{j \geq 0}$ be the eigenpair sequence of the Neumann problem in ω for the operator $-\Delta_\perp$, with $\lambda_0^{\text{neu}} = 0$ and $v_0^{\text{neu}} = 1$.
- (iv) Let $(\mu_m^{\text{dir}}, w_m^{\text{dir}})_{m \geq 1}$ be the eigenpair sequence of the Dirichlet problem in I for the operator $-\partial_3^2$.
- (v) Let $(\mu_m^{\text{neu}}, w_m^{\text{neu}})_{m \geq 0}$ be the eigenpair sequence of the Neumann problem in I for the operator $-\partial_3^2$, with $\mu_0^{\text{neu}} = 0$ and $w_0^{\text{neu}} = 1$.

One of the results of this paper is

Theorem 1.4. *Assume that ω is simply connected. Then the Maxwell eigenvalues k^2 span the set*

$$\{\lambda_j^{\text{dir}} + \mu_m^{\text{neu}}, \quad j \geq 1, m \geq 0\} \cup \{\lambda_j^{\text{neu}} + \mu_m^{\text{dir}}, \quad j \geq 1, m \geq 1\}. \quad (1.3)$$

(including repetition according to multiplicities).

In the sequel we describe a corresponding basis of eigenvectors, constructed on the model of vector wave functions, according to the widely used **M** and **N** ansatz (Debye potentials). We include the case when ω is multiply connected: In this case, the relevant parameter is the number L of connected components of $\partial\omega$ and to the set (1.3), we have to add all the μ_m^{dir} , each of them with multiplicity $L - 1$, corresponding to the number of holes contained in ω . The corresponding modes are the TEM modes that have no component in the direction x_3 .

This paper is organized as follows. In section 2 we introduce general principles for the description of the Maxwell cavity modes. In section 3 we give formulas for the electric eigenmodes $(\kappa^2, \mathbf{E})$ in the case when Ω has the cylindric form $\omega \times I$ with $\omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, separating the modes according to their polarization in TE, TM and TEM types. In section 4 we deduce the structure of magnetic cavity modes and synthesize results in Table 1. In section 5 we mention generalizations to special combinations of conducting and insulating boundary conditions.

As an application of our formulas, we consider in section 6 the case when Ω is a cube (or, more generally, a cuboid), and in section 7 the case when Ω is axisymmetric: Then Ω is a circular cylinder, or a circular cylinder with a coaxial cylindrical hole. We bring special attention to the latter case. Then the TEM modes appear in the explicit form (7.8).

We address the situation when Ω is a ball of radius R in section 8. The analysis is in the same spirit and exhibits a close relation with a scalar Laplace-like operator in the “cylinder” $\mathbb{S}^2 \times (0, R)$.

Finally, in section 9, again for product domains, we investigate the variable coefficient case, namely when ε is varying transversally, i.e. independently of the axial

variable x_3 . Then the TE and TM structures are no longer a valid Ansatz, in general. In replacement, we obtain wave guide formulations with separation of variables and tensor product form for eigenmodes.

2 Preliminaries

2.1 Electric and magnetic formulations for the Maxwell spectrum

We first recall the definition of the standard functional spaces associated with Maxwell equations on a domain $\Omega \subset \mathbb{R}^3$. The curl in 3D is defined as

$$\operatorname{curl} \mathbf{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \quad \text{for } \mathbf{u} = (u_1, u_2, u_3)$$

and $\mathbf{H}(\operatorname{curl}, \Omega)$ is the space of $L^2(\Omega)$ fields with curl in $L^2(\Omega)$, while $\mathbf{H}_0(\operatorname{curl}, \Omega)$ is the subspace of $\mathbf{H}(\operatorname{curl}, \Omega)$ with perfectly conducting electric boundary condition $\mathbf{u} \times \mathbf{n} = 0$.

The divergence in 3D is defined as

$$\operatorname{div} \mathbf{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 \quad \text{for } \mathbf{u} = (u_1, u_2, u_3)$$

and $\mathbf{H}(\operatorname{div}, \Omega)$ is the space of $L^2(\Omega)$ fields with divergence in $L^2(\Omega)$, and $\mathbf{H}_0(\operatorname{div}, \Omega)$ the subspace of $\mathbf{H}(\operatorname{div}, \Omega)$ with perfectly conducting magnetic boundary conditions $\mathbf{u} \cdot \mathbf{n} = 0$.

It is well known that the system of equations (1.1) can be formulated with $\mathbf{E}$ only (electric formulation) or $\mathbf{H}$ only (magnetic formulation). Each time a vector Helmholtz equation is found. Convenient functional spaces for the electric and magnetic variational formulations are

$$\mathbf{X}_N(\Omega) := \mathbf{H}_0(\operatorname{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega) \quad \text{and} \quad \mathbf{X}_T(\Omega) := \mathbf{H}(\operatorname{curl}, \Omega) \cap \mathbf{H}_0(\operatorname{div}, \Omega).$$

In these spaces, *regularized* formulations make sense. This means that, introducing a parameter

$$s \geq 0$$

we introduce the electric variational formulations:

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ with $\mathbf{u} \neq 0$ in $\mathbf{X}_N(\Omega)$ such that

$$\int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} + s \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \Lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{X}_N(\Omega), \quad (2.1)$$

while magnetic formulations are:

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ with $\mathbf{u} \neq 0$ in $\mathbf{X}_T(\Omega)$ such that

$$\int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} + s \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \Lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{X}_T(\Omega). \quad (2.2)$$

Relying on [4, Theorem 1.1], we know that the eigenpairs of (2.1) split in two families

- a) the Maxwell eigenvalues, independent of s , for which the eigenvectors are divergence free,
- b) the gradients of the Dirichlet eigenvectors for $-\Delta$ on Ω , associated with eigenvalues $s\lambda_{\Omega}^{\operatorname{dir}}$.

Thus the regularization by $s \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}$ makes the problem elliptic as soon as $s > 0$ and gives a description of the infinite dimensional kernel of the curl curl operator. The gauge conditions $\operatorname{div} \mathbf{E} = 0$ and $\operatorname{div} \mathbf{H} = 0$ in (1.1) ensure that we are always in case a). We can state

Lemma 2.1. (i) *Let $(k, \mathbf{E}, \mathbf{H})$ be a Maxwell eigenmode solution of (1.1). Set $\Lambda = k^2$. Then, if $\mathbf{E} \neq 0$, it is solution of (2.1) for any $s \geq 0$, and if $\mathbf{H} \neq 0$, it is solution of (2.2) for any $s \geq 0$.*

(ii) *Let $s \geq 0$. If $\Lambda \neq 0$ and $\mathbf{u}$ is solution of (2.1) with $\operatorname{div} \mathbf{u} = 0$, then setting $k = \pm\sqrt{\Lambda}$, $\mathbf{E} = \mathbf{u}$, and $\mathbf{H} = \frac{1}{ik} \operatorname{curl} \mathbf{E}$, we obtain an eigenmode of (1.1).*

(iii) *Let $s \geq 0$. If $\Lambda \neq 0$ and $\mathbf{u}$ is solution of (2.2) with $\operatorname{div} \mathbf{u} = 0$, then setting $k = \pm\sqrt{\Lambda}$, $\mathbf{H} = \mathbf{u}$, and $\mathbf{E} = -\frac{1}{ik} \operatorname{curl} \mathbf{H}$, we obtain an eigenmode of (1.1).*

2.2 Product domain

Let $\Omega \subset \mathbb{R}^3$ be of product form $\omega \times I$, with $\omega \subset \mathbb{R}^2$ and an interval I . We denote Cartesian coordinates and component of vectors as

$$x = (x_1, x_2, x_3) = (x_{\perp}, x_3) \quad \text{and} \quad \mathbf{u} = (u_1, u_2, u_3) = (\mathbf{u}_{\perp}, u_3).$$

Likewise, the exterior unit normal $\mathbf{n}$ to $\partial\Omega$ is written $(\mathbf{n}_{\perp}, n_3)$. The boundary of Ω is

$$\partial\Omega = (\partial\omega \times \bar{I}) \cup (\bar{\omega} \times \partial I).$$

On $\omega \times \partial I$, $\mathbf{n}_{\perp} = 0$ and $n_3 = \pm 1$. On $\partial\omega \times I$, $\mathbf{n}_{\perp}$ is the exterior unit normal to $\partial\omega$, $n_3 = 0$, and the tangential component of $\mathbf{u}_{\perp}$ is $\mathbf{u}_{\perp} \times \mathbf{n}_{\perp} = u_1 n_2 - u_2 n_1$. The *electric boundary conditions* $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$ are equivalent to

$$\begin{aligned} \mathbf{u}_{\perp} \times \mathbf{n}_{\perp} &= 0 \quad \text{and} \quad u_3 = 0 \quad \text{on} \quad \partial\omega \times I, \\ \mathbf{u}_{\perp} &= 0 \quad \text{on} \quad \omega \times \partial I, \end{aligned} \quad (2.3)$$

The gradient and the Laplacian in the transverse plane containing ω are denoted by $\nabla_{\perp}$ and $\Delta_{\perp}$:

$$\nabla_{\perp} v = \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix} \quad \text{and} \quad \Delta_{\perp} v = \partial_1^2 v + \partial_2^2 v.$$

The vector and scalar curls in 2D are given by:

$$\mathbf{curl}_{\perp} v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix} \quad \text{and} \quad \text{curl}_{\perp} \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

We have the formula

$$\mathbf{curl} \mathbf{u} = \begin{pmatrix} \mathbf{curl}_{\perp} u_3 \\ \text{curl}_{\perp} \mathbf{u}_{\perp} \end{pmatrix} + \partial_3 \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix}. \quad (2.4)$$

2.3 The $\mathbf{M}$, $\mathbf{N}$ ansatz and the TE or TM polarizations

The interior partial differential equation satisfied by eigenpairs is the system:

$$\mathbf{curl} \mathbf{curl} \mathbf{u} = k^2 \mathbf{u} \quad \text{and} \quad \text{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega. \quad (2.5)$$

There is a well known ansatz to solve these equations, called vector wave functions $\mathbf{M}$ and $\mathbf{N}$. They depend on the choice of a unit *piloting vector* $\hat{\mathbf{c}}$, and then $\mathbf{M}$ and $\mathbf{N}$ are generated by scalar potentials $q = q(x)$ according to

$$\mathbf{M}[q] = \mathbf{curl}(q \hat{\mathbf{c}}) \quad \text{and} \quad \mathbf{N}[q] = \mathbf{curl} \mathbf{M}[q] = \mathbf{curl} \mathbf{curl}(q \hat{\mathbf{c}}). \quad (2.6)$$

In a slightly modified form where one takes $\hat{\mathbf{c}} = \frac{x}{|x|}$, the ansatz $\mathbf{M}$ and $\mathbf{N}$ are the cornerstone for the construction of spherical wave functions, cf. section 8.

For our study, we choose

$$\hat{\mathbf{c}} = \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.7)$$

Direct calculations yield:

Lemma 2.2. *Let $q \in H^1(\Omega)$ and set $\mathbf{M}[q] = \mathbf{curl}(q \mathbf{e}_3)$. Then*

$$\mathbf{M}[q] = \begin{pmatrix} \mathbf{curl}_{\perp} q \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{curl} \mathbf{M}[q] = \nabla(\partial_3 q) - \Delta q \mathbf{e}_3. \quad (2.8)$$

With $\mathbf{N}[q] = \mathbf{curl} \mathbf{M}[q]$, we have

$$\mathbf{N}[q] = \nabla(\partial_3 q) - \Delta q \mathbf{e}_3 \quad \text{and} \quad \mathbf{curl} \mathbf{N}[q] = -\mathbf{M}[\Delta q]. \quad (2.9)$$

The form of $\mathbf{M}$ and $\text{curl } \mathbf{N}$ with their third component zero explains why $\mathbf{M}$, when describing an electric field, represents the TE (transverse electric) polarization, and $\mathbf{N}$, the TM (transverse magnetic) polarization. For the description of a magnetic field, the converse happens: $\mathbf{M}$ is TM and $\mathbf{N}$ is TE.

As a consequence, we find that

$$\begin{aligned} (\text{curl curl} - k^2)\mathbf{M}[q] &= -\mathbf{M}[\Delta q + k^2 q], \\ (\text{curl curl} - k^2)\mathbf{N}[q] &= -\mathbf{N}[\Delta q + k^2 q]. \end{aligned} \quad (2.10)$$

Thus, looking for solutions of (2.5) amounts to considering $\mathbf{M}[q]$ and $\mathbf{N}[q]$ with q solution of the Helmholtz equation $\Delta q + \kappa^2 q = 0$.

3 Electric eigenmodes in a product domain

In this section, we look for solutions $(k^2, \mathbf{E})$ of the electric problem (2.1) with the gauge constraint $\text{div } \mathbf{E} = 0$. For this we use the $\mathbf{M}, \mathbf{N}$ ansatz, we find sufficient conditions on the potentials q , construct families of eigenpairs and prove that this system is complete.

3.1 TE modes

Let $\mathbf{E} = \mathbf{M}[q]$ be a TE mode. By construction $\text{div } \mathbf{E} = 0$. By (2.10), q has to satisfy

$$\Delta q + \kappa^2 q = 0. \quad (3.1)$$

It remains to verify the electric boundary conditions $\mathbf{E} \times \mathbf{n} = 0$ on $\partial\Omega$. Combining (2.3) and (2.8), we find

$$\begin{aligned} \text{curl}_\perp q \times \mathbf{n}_\perp &= 0 \quad \text{on} \quad \partial\omega \times I, \\ \text{curl}_\perp q &= 0 \quad \text{on} \quad \omega \times \partial I, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \partial_n q &= 0 \quad \text{on} \quad \partial\omega \times I, \\ \nabla_\perp q &= 0 \quad \text{on} \quad \omega \times \partial I. \end{aligned} \quad (3.2)$$

Sufficient conditions for this are Dirichlet conditions on $\omega \times \partial I$ combined with Neumann conditions on $\partial\omega \times I$. This is a tensor product of a Neumann problem on ω and a Dirichlet problem on I . Along the same principle than for pure Dirichlet problem, cf Theorem 1.1, we find a spectral basis for q in the form

$$q_{jm} = v_j^{\text{neu}} \otimes w_m^{\text{dir}}, \quad k^2 = \lambda_j^{\text{neu}} + \mu_m^{\text{dir}}, \quad j \geq 1, m \geq 1. \quad (3.3)$$

Here $j = 0$ (corresponding to $v_0^{\text{neu}} = 1$) is discarded because functions q independent of $x_\perp$ give $\mathbf{M}[q] = 0$.

Thus we have found the following families of TE modes:

Lemma 3.1. *For all $j \geq 1$, $m \geq 1$, the field $\mathbf{E}_{jm}^{\text{TE}} := \mathbf{M}[v_j^{\text{neu}} \otimes w_m^{\text{dir}}]$, i.e.*

$$\mathbf{E}_{jm}^{\text{TE}}(x_{\perp}, x_3) = \begin{pmatrix} \mathbf{curl}_{\perp} v_j^{\text{neu}}(x_{\perp}) \\ 0 \end{pmatrix} w_m^{\text{dir}}(x_3), \quad (3.4)$$

is a TE mode for problem (2.1) associated with the eigenvalue $\Lambda_{jm}^{\text{TE}} = \lambda_j^{\text{neu}} + \mu_m^{\text{dir}}$.

3.2 TM modes

Let $\mathbf{E} = \mathbf{N}[q]$ be a TM mode. Again, $\text{div } \mathbf{E} = 0$, q has to satisfy (3.1), and it remains to verify the electric boundary conditions $\mathbf{E} \times \mathbf{n} = 0$ on $\partial\Omega$: Using (2.8), we find that

$$\mathbf{E}_{\perp} = \nabla_{\perp}(\partial_3 q) \quad \text{and} \quad E_3 = -\Delta_{\perp} q$$

Hence, with (2.3)

$$\begin{aligned} \nabla_{\perp}(\partial_3 q) \times \mathbf{n}_{\perp} &= 0 \quad \text{and} \quad \Delta_{\perp} q = 0 \quad \text{on} \quad \partial\omega \times I, \\ \nabla_{\perp}(\partial_3 q) &= 0 \quad \text{on} \quad \omega \times \partial I. \end{aligned}$$

We obtain sufficient conditions through the separation of variable ansatz

$$q(x) = v(x_{\perp}) w(x_3)$$

with

$$-\Delta_{\perp} v = \lambda v \quad \text{in } \omega \quad \text{and} \quad -\partial_3^2 w = \mu w \quad \text{in } I \quad \text{with} \quad \lambda + \mu = k^2 = \Lambda, \quad (3.5)$$

and the boundary conditions become

$$\begin{cases} (\mathbf{n}_{\perp} \times \nabla_{\perp})v(x_{\perp}) \partial_3 w(x_3) = 0 & \forall x_{\perp} \in \partial\omega, \forall x_3 \in I, \\ \Delta_{\perp} v(x_{\perp}) w(x_3) = 0 & \forall x_{\perp} \in \partial\omega, \forall x_3 \in I, \\ \nabla_{\perp} v(x_{\perp}) \partial_3 w(x_3) = 0 & \forall x_{\perp} \in \omega, \forall x_3 \in \partial I, \end{cases}$$

which yields

$$\begin{cases} v = \text{const. on each } \partial_l \omega & \text{or} \quad \partial_3 w \equiv 0 \text{ in } I, \\ \Delta_{\perp} v = 0 \text{ on } \partial\omega & \text{or} \quad w \equiv 0 \text{ in } I, \\ \nabla_{\perp} v \equiv 0 \text{ in } \omega & \text{or} \quad \partial_3 w = 0 \text{ on } \partial I. \end{cases} \quad (3.6)$$

Here, $\partial_l \omega$, $l = 1, \dots, L$, are the connected components of $\partial\omega$.

The conditions $\nabla_{\perp} v \equiv 0$ and $w \equiv 0$ have to be discarded since they imply $\mathbf{E} \equiv 0$. Therefore we should have $\partial_3 w = 0$ on ∂I and $\Delta_{\perp} v = 0$ on $\partial\omega$. The latter condition implies that $v = 0$ on $\partial\omega$ in the case when $\lambda \neq 0$. When $\lambda = 0$, the condition

$v = \text{const.}$ on each $\partial_l \omega$ is sufficient. Thus we have shown that (3.5)-(3.6) can be summarized as follows: Either

$$\begin{cases} -\Delta_{\perp} v = \lambda v \text{ in } \omega & \text{and } v = 0 \text{ on } \partial\omega \\ -\partial_3^2 w = \mu w \text{ in } I & \text{and } \partial_3 w = 0 \text{ on } \partial I \end{cases} \quad \text{with } \lambda \neq 0, \lambda + \mu = \Lambda, \quad (3.7)$$

or

$$\begin{cases} -\Delta_{\perp} v = 0 \text{ in } \omega & \text{and } v = \text{const on each } \partial_l \omega \\ -\partial_3^2 w = \mu w \text{ in } I & \text{and } \partial_3 w = 0 \text{ on } \partial I \end{cases} \quad \text{with } \mu = \Lambda. \quad (3.8)$$

Hence we have found the following two families of TM modes. First, the standard one:

Lemma 3.2. *For all $j \geq 1$, $m \geq 0$, the field $\mathbf{E}_{jm}^{\text{TM}} := \mathbf{N}[v_j^{\text{dir}} \otimes w_m^{\text{neu}}]$, i.e.*

$$\mathbf{E}_{jm}^{\text{TM}}(x_{\perp}, x_3) = \begin{pmatrix} \nabla_{\perp} v_j^{\text{dir}}(x_{\perp}) \\ 0 \end{pmatrix} \partial_3 w_m^{\text{neu}}(x_3) - \begin{pmatrix} 0 \\ \Delta_{\perp} v_j^{\text{dir}}(x_{\perp}) \end{pmatrix} w_m^{\text{neu}}(x_3), \quad (3.9)$$

is a TM mode for problem (2.1) associated with the eigenvalue $\Lambda_{jm}^{\text{TM}} = \lambda_j^{\text{dir}} + \mu_m^{\text{neu}}$.

The second family appears if ω has a non trivial topology, and shares the features of TE and TM polarization:

Lemma 3.3. *If $\partial\omega$ has more than one connected components ($L \geq 2$), there exist $L - 1$ independent, non-constant, harmonic potentials v_l^{top} , $l = 1, \dots, L - 1$ with constant traces on each connected component of $\partial\omega$. They generate the family of TEM modes defined for all $l = 1, \dots, L - 1$ and $m \geq 1$ as the fields $\mathbf{E}_{lm}^{\text{TEM}} := \mathbf{N}[v_l^{\text{top}} \otimes w_m^{\text{neu}}]$ which can also be written as*

$$\mathbf{E}_{lm}^{\text{TEM}}(x_{\perp}, x_3) = \begin{pmatrix} \nabla_{\perp} v_l^{\text{top}}(x_{\perp}) \\ 0 \end{pmatrix} w_m^{\text{neu}}(x_3), \quad (3.10)$$

and is associated with the eigenvalue $\Lambda_{lm}^{\text{TEM}} = \mu_m^{\text{dir}}$.

Note that to obtain (3.10) we have used that the derivatives $\partial_3 w_k^{\text{neu}}$ for $k \geq 1$ are an eigenvector basis for the Dirichlet problem on the interval I .

Remark 3.4. Let us borrow the following objects from [1]: Let ω° be $\omega \setminus \Sigma$, where $\Sigma = \cup_{l=1}^{L-1} \Sigma_l$ is a minimal set of cuts so that ω° is simply connected. Then we can define the space $\Theta(\omega)$ as:

$$\Theta(\omega) = \{\varphi \in H^1(\omega^{\circ}) \mid [\varphi]_{\Sigma_l} = \text{const}(l), \quad l = 1, \dots, L - 1\}.$$

For $\varphi \in \Theta(\omega)$, its extended $\mathbf{curl}_\perp$ denoted by $\widetilde{\mathbf{curl}}_\perp \varphi$ is defined as its $\mathbf{curl}_\perp$ in ω° , considered as an element of $L^2(\omega)$. Then there exist “conjugate” potentials $\tilde{v}_l^{\text{top}} \in \Theta(\omega)$ such that for any $l \leq L - 1$, there holds

$$\widetilde{\mathbf{curl}}_\perp \tilde{v}_l^{\text{top}} = \nabla_\perp v_l^{\text{top}}. \quad (3.11)$$

Therefore for all $m \geq 1$, the mode $\mathbf{E}_{lm}^{\text{TEM}}$ is also an extended TE mode. This is why it is called a TEM mode.

3.3 Completeness

The aim of this section is to prove

Lemma 3.5. *Let $\mathbf{u} \in \mathbf{X}_N(\Omega)$ such that $\text{div } \mathbf{u} = 0$. We assume that for all integers $j \geq 1$ and $l \in [1, L - 1]$*

$$\langle \mathbf{u}, \mathbf{E}_{jm}^{\text{TE}} \rangle = 0 \ (\forall m \geq 1), \quad \langle \mathbf{u}, \mathbf{E}_{jm}^{\text{TM}} \rangle = 0 \ (\forall m \geq 0) \quad \text{and} \quad \langle \mathbf{u}, \mathbf{E}_{lm}^{\text{TEM}} \rangle = 0 \ (\forall m \geq 1).$$

Here $\langle \cdot, \cdot \rangle$ is the L^2 scalar product on Ω . Then $\mathbf{u} = 0$.

Proof. We first draw consequences from the orthogonality properties against the TM modes: We fix j and m and set $v = v_j^{\text{dir}}$, $w = w_m^{\text{neu}}$ and integrate by parts:

$$\begin{aligned} 0 &= \int_I \int_\omega \mathbf{u}_\perp(x_\perp, x_3) \nabla_\perp v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3) \Delta_\perp v(x_\perp) w(x_3) \, dx_\perp dx_3 \\ &= \int_I \int_\omega -\text{div}_\perp \mathbf{u}_\perp(x_\perp, x_3) v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3) \Delta_\perp v(x_\perp) w(x_3) \, dx_\perp dx_3 \\ &= \int_I \int_\omega \partial_3 u_3(x_\perp, x_3) v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3) \Delta_\perp v(x_\perp) w(x_3) \, dx_\perp dx_3 \\ &= \int_I \int_\omega -u_3(x_\perp, x_3) v(x_\perp) \partial_3^2 w(x_3) - u_3(x_\perp, x_3) \Delta_\perp v(x_\perp) w(x_3) \, dx_\perp dx_3. \end{aligned}$$

Here we have used that $\text{div } \mathbf{u} = 0$, replacing $\text{div}_\perp \mathbf{u}_\perp$ by $-\partial_3 u_3$. Coming back to the properties of $v = v_j^{\text{dir}}$ and $w = w_m^{\text{neu}}$ we find for all $j \geq 1$ and $m \geq 0$

$$\int_I \int_\omega u_3(x_\perp, x_3) (\lambda_j^{\text{dir}} + \mu_m^{\text{neu}}) v_j^{\text{dir}}(x_\perp) w_m^{\text{neu}}(x_3) \, dx_\perp dx_3 = 0.$$

Since $\lambda_j^{\text{dir}} + \mu_m^{\text{neu}}$ is never 0, we deduce that for all $j \geq 1$ and $m \geq 0$

$$\int_I \int_\omega u_3(x_\perp, x_3) v_j^{\text{dir}}(x_\perp) w_m^{\text{neu}}(x_3) \, dx_\perp dx_3 = 0.$$

The set $v_j^{\text{dir}}(x_\perp) w_m^{\text{neu}}(x_3)$ being a complete basis in $L^2(\Omega)$, we deduce that $u_3 = 0$.

Next, we use the orthogonality against the TE modes: for all $j \geq 1$ and $m \geq 1$ there holds:

$$\int_I w_m^{\text{dir}}(x_3) \int_{\omega} \mathbf{u}_{\perp}(x_{\perp}, x_3) \cdot \mathbf{curl}_{\perp} v_j^{\text{neu}}(x_{\perp}) \, dx_{\perp} dx_3 = 0.$$

Therefore, for all $j \geq 1$:

$$\int_{\omega} \mathbf{u}_{\perp}(x_{\perp}, x_3) \cdot \mathbf{curl}_{\perp} v_j^{\text{neu}}(x_{\perp}) \, dx_{\perp} = 0, \quad \forall x_3 \in I.$$

We deduce that $\mathbf{curl}_{\perp} \mathbf{u}_{\perp}(\cdot, x_3)$ is orthogonal to all v_j^{neu} for $j \geq 1$, which means that $\mathbf{curl}_{\perp} \mathbf{u}_{\perp}(\cdot, x_3)$ is constant with respect to $x_{\perp}$. There exists a function $z = z(x_3)$ such that

$$(*) \quad \mathbf{curl}_{\perp} \mathbf{u}_{\perp}(x_{\perp}, x_3) = z(x_3).$$

Since $\text{div} \mathbf{u} = 0$ and $u_3 = 0$, we have $\text{div}_{\perp} \mathbf{u}_{\perp} = 0$. Besides, the orthogonality relations against the TEM modes yields for all $m \geq 1$ and $l \leq L - 1$

$$\int_I w_m^{\text{dir}}(x_3) \int_{\omega} \mathbf{u}_{\perp}(x_{\perp}, x_3) \cdot \nabla_{\perp} v_l^{\text{top}}(x_{\perp}) \, dx_{\perp} dx_3 = 0.$$

We deduce that

$$\int_{\omega} \mathbf{u}_{\perp}(x_{\perp}, x_3) \cdot \nabla_{\perp} v_l^{\text{top}}(x_{\perp}) \, dx_{\perp} = 0, \quad \forall x_3 \in I,$$

from which we find that

$$\int_{\partial\omega_l} \mathbf{u}_{\perp} \cdot \mathbf{n}_{\perp} \, d\sigma = 0, \quad l = 1, \dots, L.$$

Combined with $\text{div}_{\perp} \mathbf{u}_{\perp} = 0$, this provides the existence of a potential $y \in L^2(I, H^1(\omega))$ satisfying the Neumann boundary condition on $\partial\omega$ such that

$$\mathbf{u}_{\perp}(x_{\perp}, x_3) = \mathbf{curl}_{\perp} y(x_{\perp}, x_3).$$

With $(*)$ we find

$$-\Delta_{\perp} y(x_{\perp}, x_3) = z(x_3).$$

Since y satisfies the homogeneous Neumann condition with respect to $x_{\perp}$, this implies that $z(x_3) = 0$ for all x_3 . Finally we have obtained that $\mathbf{u}_{\perp} = 0$. $\square$

3.4 Eigenmodes

Summarizing, we have proved:

Theorem 3.6. *Let $\Omega = \omega \times I$. The eigenpairs with zero divergence of the electric Maxwell operator (2.1) can be organized in the three families:*

$$\begin{aligned} \mathbf{E}_{jm}^{\text{TE}} &= \begin{pmatrix} \text{curl}_{\perp} v_j^{\text{neu}}(x_{\perp}) \\ 0 \end{pmatrix} w_m^{\text{dir}}(x_3) \quad \text{with} \quad \Lambda_{jm}^{\text{TE}} = \lambda_j^{\text{neu}} + \mu_m^{\text{dir}}, \quad j \geq 1, m \geq 1, \\ \mathbf{E}_{jm}^{\text{TM}} &= \begin{pmatrix} \nabla_{\perp} v_j^{\text{dir}}(x_{\perp}) \\ 0 \end{pmatrix} \partial_3 w_m^{\text{neu}}(x_3) - \begin{pmatrix} 0 \\ \Delta_{\perp} v_j^{\text{dir}}(x_{\perp}) \end{pmatrix} w_m^{\text{neu}}(x_3) \\ &\quad \text{with} \quad \Lambda_{jm}^{\text{TM}} = \lambda_j^{\text{dir}} + \mu_m^{\text{neu}}, \quad j \geq 1, m \geq 0, \\ \mathbf{E}_{lm}^{\text{TEM}} &= \begin{pmatrix} \nabla_{\perp} v_l^{\text{top}}(x_{\perp}) \\ 0 \end{pmatrix} w_m^{\text{dir}}(x_3) \quad \text{with} \quad \Lambda_{lm}^{\text{TEM}} = \mu_m^{\text{dir}}, \quad 1 \leq l \leq L-1, m \geq 1. \end{aligned}$$

See Notation 1.3 and Lemma 3.2 for the notations of the 2D and 1D quantities. All the associated eigenvalues Λ_{jm}^{TE} , Λ_{jm}^{TM} and $\Lambda_{lm}^{\text{TEM}}$ are non-zero.

4 Magnetic eigenmodes in a product domain

Since the magnetic field $\mathbf{H}$ associated with the electric field $\mathbf{E}$ is given by

$$\mathbf{H} = \frac{1}{ik} \text{curl } \mathbf{E}, \quad \text{for } k = \pm\sqrt{\Lambda}$$

for any non-zero eigenvalue Λ , we deduce:

Corollary 4.1. *Under the conditions of Theorem 3.6, we set $k = \pm\sqrt{\Lambda}$. The associated magnetic fields are given by*

$$\begin{aligned} \mathbf{H}_{jm}^{\text{TE}} &= \frac{1}{ik_{jm}^{\text{TE}}} \left\{ \begin{pmatrix} \nabla_{\perp} v_j^{\text{neu}}(x_{\perp}) \\ 0 \end{pmatrix} \partial_3 w_m^{\text{dir}}(x_3) - \begin{pmatrix} 0 \\ \Delta_{\perp} v_j^{\text{neu}}(x_{\perp}) \end{pmatrix} w_m^{\text{dir}}(x_3) \right\} \quad j, m \geq 1, \\ \mathbf{H}_{jm}^{\text{TM}} &= -ik_{jm}^{\text{TM}} \begin{pmatrix} \text{curl}_{\perp} v_j^{\text{dir}}(x_{\perp}) \\ 0 \end{pmatrix} w_m^{\text{neu}}(x_3) \quad j \geq 1, m \geq 0, \\ \mathbf{H}_{lm}^{\text{TEM}} &= \frac{i}{k_{lm}^{\text{TEM}}} \begin{pmatrix} \text{curl}_{\perp} v_l^{\text{top}}(x_{\perp}) \\ 0 \end{pmatrix} \partial_3 w_m^{\text{dir}}(x_3) \quad 1 \leq l \leq L-1, m \geq 1, \end{aligned}$$

and the triples $(k, \mathbf{E}, \mathbf{H})$ are Maxwell eigenmodes.

Remark 4.2. (i) The electric fields in the pairs $(\mathbf{E}^{\text{TE}}, \mathbf{H}^{\text{TE}})$ are transverse to the axis x_3 , whilst in the pairs $(\mathbf{E}^{\text{TM}}, \mathbf{H}^{\text{TM}})$ the magnetic fields are transverse to the axis x_3 , which justifies the labels of the polarizations.

(ii) We notice that for all $m \geq 1$, $\mathbf{H}_{lm}^{\text{TEM}}$ can also be written as

$$\mathbf{H}_{lm}^{\text{TEM}} = i \begin{pmatrix} \mathbf{curl}_{\perp} v_l^{\text{top}}(x_{\perp}) \\ 0 \end{pmatrix} w_m^{\text{neu}}(x_3)$$

The expression above also makes sense for $m = 0$. The associated eigenvalue is 0 and the corresponding electric field is 0. These magnetostatic Maxwell eigenmodes $(0, 0, \mathbf{H}_{l0}^{\text{TEM}})$ are those produced by the 3D topological non-triviality of Ω .

Remark 4.3. If ω contains holes, i.e. if TEM modes are present, they often contribute the smallest positive eigenvalues. Let us make formulas for eigenvalues more explicit: Let ℓ be the length of the interval I and let us assume that ω has *one hole*. Besides the magnetostatic zero eigenvalue, we find

$$\Lambda_{jm}^{\text{TE}} = \lambda_j^{\text{neu}} + \left(\frac{m\pi}{\ell}\right)^2 \quad (\forall j, m \geq 1), \quad \Lambda_{jm}^{\text{TM}} = \lambda_j^{\text{dir}} + \left(\frac{m\pi}{\ell}\right)^2 \quad (\forall j \geq 1, m \geq 0),$$

and

$$\Lambda_m^{\text{TEM}} = \left(\frac{m\pi}{\ell}\right)^2 \quad (\forall m \geq 1).$$

Then the smallest positive eigenvalue is either $\Lambda_{1,0}^{\text{TM}}$ or Λ_1^{TEM} . If ω is fixed and ℓ large enough, Λ_1^{TEM} is smaller than $\Lambda_{1,0}^{\text{TM}}$.

We summarize the results of sections 3 and 4 in Table 1.

Polarization	k^2	$\mathbf{E}$	$\mathbf{H}$
TE	$\lambda_j^{\text{neu}} + \left(\frac{m\pi}{\ell}\right)^2$	$\mathbf{M}[v_j^{\text{neu}} \otimes \sin(\frac{m\pi}{\ell} \cdot)]$	$\frac{1}{ik} \mathbf{N}[v_j^{\text{neu}} \otimes \sin(\frac{m\pi}{\ell} \cdot)]$
TM	$\lambda_j^{\text{dir}} + \left(\frac{m\pi}{\ell}\right)^2$	$\mathbf{N}[v_j^{\text{dir}} \otimes \cos(\frac{m\pi}{\ell} \cdot)]$	$ik \mathbf{M}[v_j^{\text{dir}} \otimes \cos(\frac{m\pi}{\ell} \cdot)]$
TEM	$\left(\frac{m\pi}{\ell}\right)^2$	$\mathbf{N}[v_l^{\text{top}} \otimes \cos(\frac{m\pi}{\ell} \cdot)]$	$ik \mathbf{M}[v_l^{\text{top}} \otimes \cos(\frac{m\pi}{\ell} \cdot)]$
Magnetostatic	0	0	$\mathbf{M}[v_l^{\text{top}} \otimes 1]$

Table 1. Synthetic description of Maxwell eigenmodes, using $\mathbf{M}$ and $\mathbf{N}$ (2.6).

5 Mixed perfectly conducting or insulating conditions

Consider now the situation where a part $\partial\Omega_{\text{cd}}$ of the boundary of Ω represents *perfectly conducting walls* whereas another part $\partial\Omega_{\text{ins}}$ represents *perfectly insulating walls*, with

$$\partial\Omega = \partial\Omega_{\text{cd}} \cup \partial\Omega_{\text{ins}}, \quad \partial\Omega_{\text{cd}} \cap \partial\Omega_{\text{ins}} = \emptyset. \quad (5.1)$$

Boundary conditions are then

$$\begin{cases} \mathbf{E} \times \mathbf{n} = 0 & \text{and} & \mathbf{H} \cdot \mathbf{n} = 0, & \text{on} & \partial\Omega_{\text{cd}}, & (\text{perfect conductor b. c.}) \\ \mathbf{E} \cdot \mathbf{n} = 0 & \text{and} & \mathbf{H} \times \mathbf{n} = 0, & \text{on} & \partial\Omega_{\text{ins}}, & (\text{perfect insulator b. c.}) \end{cases}$$

Similar results as above hold for *mixed boundary conditions* when the perfectly conducting or insulating parts $\partial\Omega_{\text{cd}}$ and $\partial\Omega_{\text{ins}}$ are chosen to be either $\partial\omega \times I$ or $\omega \times \partial I$. Let us give two examples.

Example 5.1. Let us consider the case when

$$\partial\Omega_{\text{cd}} = \partial\omega \times I \quad \text{and} \quad \partial\Omega_{\text{ins}} = \omega \times \partial I.$$

Then, the essential boundary condition for the electric field $\mathbf{E}$ on $\omega \times \partial I$ is $\mathbf{E}_3 = 0$ and the natural boundary condition is $\text{curl } \mathbf{E} \times \mathbf{n} = 0$, reducing to $\partial_3 \mathbf{E}_\perp = 0$. Thus we find the three families of electric eigenfunctions:

$$\begin{aligned} \mathbf{E}_{jm}^{\text{TE}} &= \begin{pmatrix} \text{curl}_\perp v_j^{\text{neu}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\text{neu}}(x_3) \quad \text{with } j \geq 1, m \geq 0, \\ \mathbf{E}_{jm}^{\text{TM}} &= \begin{pmatrix} \nabla_\perp v_j^{\text{dir}}(x_\perp) \\ 0 \end{pmatrix} \partial_3 w_m^{\text{dir}}(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v_j^{\text{dir}}(x_\perp) \end{pmatrix} w_m^{\text{dir}}(x_3), \quad \text{with } j \geq 1, m \geq 1, \\ \mathbf{E}_{lm}^{\text{TEM}} &= \begin{pmatrix} \nabla_\perp v_l^{\text{top}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\text{neu}}(x_3) \quad \text{with } 1 \leq l \leq L-1, m \geq 0. \end{aligned}$$

associated with the eigenvalues $\Lambda_{jm}^{\text{TE}} = \lambda_j^{\text{neu}} + \mu_m^{\text{neu}}$, $\Lambda_{jm}^{\text{TM}} = \lambda_j^{\text{dir}} + \mu_m^{\text{dir}}$, and $\Lambda_{lm}^{\text{TEM}} = \mu_m^{\text{neu}}$.

Example 5.2. We set $I = (0, \ell)$. Let us consider the case when

$$\partial\Omega_{\text{cd}} = (\partial\omega \times I) \cup (\omega \times \{0\}) \quad \text{and} \quad \partial\Omega_{\text{ins}} = \omega \times \{\ell\}.$$

The axial generators w_m can be described thanks to the eigenvectors w_m^{mix} , $m \geq 1$, of the *mixed* problem in ω :

$$-\partial_3^2 w = \mu w, \quad w(0) = 0, \quad \partial_3 w(\ell) = 0.$$

We find

$$\begin{aligned} \mathbf{E}_{jm}^{\text{TE}} &= \begin{pmatrix} \text{curl}_\perp v_j^{\text{neu}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\text{mix}}(x_3) \quad \text{with } j \geq 1, m \geq 1, \\ \mathbf{E}_{jm}^{\text{TM}} &= \begin{pmatrix} \nabla_\perp v_j^{\text{dir}}(x_\perp) \\ 0 \end{pmatrix} \partial_3^2 w_m^{\text{mix}}(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v_j^{\text{dir}}(x_\perp) \end{pmatrix} \partial_3 w_m^{\text{mix}}(x_3), \quad \text{with } j \geq 1, m \geq 1, \\ \mathbf{E}_{lm}^{\text{TEM}} &= \begin{pmatrix} \nabla_\perp v_l^{\text{top}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\text{mix}}(x_3) \quad \text{with } 1 \leq l \leq L-1, m \geq 1. \end{aligned}$$

If ω contains holes, TEM modes are present and contribute the smallest positive eigenvalue $\left(\frac{\pi}{2\ell}\right)^2$.

6 Application 1: Maxwell eigenvalues of cuboids

6.1 Cube

Let Ω be the cube $(0, \pi)^3$. We can apply Theorem 3.6 with $\omega = (0, \pi)^2$ and $I = (0, \pi)$. Since ω is simply connected we have TE and TM modes only. Therefore the normalized Maxwell eigenvalues are

$$\lambda_j^{\text{neu}} + \mu_m^{\text{dir}}, \quad j \geq 1, \quad m \geq 1 \quad \text{and} \quad \lambda_j^{\text{dir}} + \mu_m^{\text{neu}}, \quad j \geq 1, \quad m \geq 0.$$

We have

$$\mu_m^{\text{dir}} = m^2, \quad m \geq 1 \quad \text{and} \quad \mu_m^{\text{neu}} = m^2, \quad m \geq 0.$$

The Dirichlet eigenvalues on ω are

$$k_1^2 + k_2^2, \quad k_1, k_2 \geq 1.$$

The non-zero Neumann eigenvalues are

$$k_1^2 + k_2^2, \quad k_1, k_2 \geq 0, \quad k_1 \text{ or } k_2 \neq 0.$$

Therefore the TE eigenvalues are

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2 \geq 0, \quad k_1 \text{ or } k_2 \neq 0, \quad k_3 \geq 1.$$

The TM eigenvalues are

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2 \geq 1, \quad k_3 \geq 0.$$

Therefore we have once

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2, k_3 \geq 0 \quad \text{with one index } \nu \in \{1, 2, 3\} \text{ such that } k_\nu = 0,$$

and twice

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2, k_3 \geq 1.$$

The first eigenvalues are

$$2 \text{ (mult. 3)}, \quad 3 \text{ (mult. 2)}, \quad 5 \text{ (mult. 6)}, \quad 6 \text{ (mult. 6)}, \quad 8 \text{ (mult. 3)}, \dots$$

A larger multiplicity of 12 is attained for example for $14 = 1 + 4 + 9$. But 12 is not the maximal multiplicity (e.g. the multiplicity of $26 = 25 + 1 + 0 = 16 + 9 + 1$ is 18).

The Dirichlet eigenvectors on $(0, \pi)$ are $\zeta \mapsto \sin k\zeta$, $k \geq 1$, and the Neumann eigenvectors are $\cos k\zeta$, $k \geq 0$. The components of the electric eigenvectors in the cube are (sums of) products of two sin terms by one cos term.

6.2 Cuboids

For a *rectangular parallelepiped*

$$\Omega = (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3),$$

we find the eigenvalues: Once

$$\left(\frac{k_1\pi}{\ell_1}\right)^2 + \left(\frac{k_2\pi}{\ell_2}\right)^2 + \left(\frac{k_3\pi}{\ell_3}\right)^2, \\ \forall k_1, k_2, k_3 \geq 0 \quad \text{with one index } \nu \in \{1, 2, 3\} \text{ such that } k_\nu = 0,$$

and twice

$$\left(\frac{k_1\pi}{\ell_1}\right)^2 + \left(\frac{k_2\pi}{\ell_2}\right)^2 + \left(\frac{k_3\pi}{\ell_3}\right)^2, \quad \forall k_1, k_2, k_3 \geq 1.$$

7 Application 2: Maxwell eigenvalues in axisymmetric product domains

We assume now, besides the assumption that $\Omega = \omega \times I$, that the domain Ω is axisymmetric. This implies that ω is an axisymmetric domain in dimension 2. Hence ω is either a disc or an annulus (i.e. a disc with a concentric hole). We investigate both situations.

7.1 Axisymmetric domains

Let R be the external radius of ω and r_0 be its internal radius, with the convention that $r_0 = 0$ corresponds to the case when ω is a disc. Let us denote by $\mathbb{T}$ the one-dimensional torus

$$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}).$$

We use *cylindrical coordinates* $(r, \varphi, x_3) \in (r_0, R) \times \mathbb{T} \times I$. Setting

$$\check{u}(r, \varphi, x_3) = u(x),$$

we introduce *cylindrical components* (u_r, u_φ, u_3) of the field $\mathbf{u} = (u_1, u_2, u_3)$,

$$u_r = \check{u}_1 \cos \varphi + \check{u}_2 \sin \varphi \quad \text{and} \quad u_\varphi = -\check{u}_1 \sin \varphi + \check{u}_2 \cos \varphi.$$

In particular, for a scalar function q , the radial and angular components of $\nabla_\perp q$ are $\partial_r q$ and $\frac{1}{r}\partial_\varphi q$, and those of $\mathbf{curl}_\perp q$ are $\frac{1}{r}\partial_\varphi q$ and $-\partial_r q$. With this, we find the representation in cylindrical coordinates of the ansatz $\mathbf{M}[q]$ and $\mathbf{N}[q]$ when q has the tensor form $v \otimes w$:

$$\begin{cases} \mathbf{M}_r[v \otimes w] = \frac{1}{r}\partial_\varphi v(r, \varphi) w(x_3), \\ \mathbf{M}_\varphi[v \otimes w] = -\partial_r v(r, \varphi) w(x_3), \\ \mathbf{M}_3[v \otimes w] = 0, \end{cases} \quad (7.1)$$

and

$$\begin{cases} \mathbf{N}_r[v \otimes w] = \partial_r v(r, \varphi) \partial_3 w(x_3), \\ \mathbf{N}_\varphi[v \otimes w] = \frac{1}{r} \partial_\varphi v(r, \varphi) \partial_3 w(x_3), \\ \mathbf{N}_3[v \otimes w] = -\frac{1}{r^2} ((r \partial_r)^2 + \partial_\varphi^2) v(r, \varphi) w(x_3). \end{cases} \quad (7.2)$$

To describe the Maxwell eigenmodes in the axisymmetric case, we use Table 1 and make explicit the Dirichlet and Neumann eigenvectors v^{dir} and v^{neu} on ω , and also v^{top} when there is a hole ($r_0 > 0$).

It is a classical technique to use the invariance under rotation of the Laplace operator $\Delta_\perp$ for diagonalizing it by Fourier series with respect to $\varphi \in \mathbb{T}$. This leads to the following representations:

$$v^{\text{dir}} = h_{np}^{\text{dir}}(r) e^{in\varphi} \quad \text{and} \quad v^{\text{neu}} = h_{np}^{\text{neu}}(r) e^{in\varphi}, \quad n \in \mathbb{Z}, \quad p \geq 1. \quad (7.3)$$

where for each $n \in \mathbb{Z}$, the functions $(h_{np}^{\text{dir}})_p$ and $(h_{np}^{\text{neu}})_p$ are bases of eigenfunctions for the operator

$$h \mapsto \left(-\partial_r^2 - \frac{1}{r} \partial_r + \frac{n^2}{r^2} \right) h, \quad r \in (r_0, R) \quad (7.4)$$

with appropriate boundary conditions.

7.2 The cylinder (ω is a disc)

For h^{dir} the boundary condition at R is $h^{\text{dir}}(R) = 0$, for h^{neu} this is $\partial_r h^{\text{neu}}(R) = 0$. At the other end $r = 0$ of the interval $(0, R)$, the boundary conditions are driven by integrability properties, cf [3]: For h^{dir} and h^{neu} , they are

$$\partial_r h(0) = 0 \quad \text{if} \quad n = 0, \quad \text{and} \quad h(0) = 0 \quad \text{if} \quad n \neq 0, \quad (7.5)$$

As a consequence, both h^{dir} and h^{neu} are given by the Bessel functions of the first kind J_n that satisfy (7.5) and the equation $(-\partial_r^2 - \frac{1}{r} \partial_r + n^2) J_n = J_n$, cf (7.4). One finds

Lemma 7.1 ([6]). (i) Let $(z_{np})_{p \geq 1}$ be the increasing sequence of the positive zeros of J_n . Then a spectral sequence for the Dirichlet problem for $-\Delta_\perp$ on ω is

$$\lambda_{np}^{\text{dir}} = \left(\frac{z_{np}}{R} \right)^2 \quad \text{and} \quad v_{np}^{\text{dir}} = J_n \left(\frac{z_{np} r}{R} \right) e^{in\varphi}, \quad n \in \mathbb{Z}, \quad p \geq 1 \quad (7.6)$$

(ii) Let $(z'_{np})_{p \geq 1}$ be the increasing sequence of the positive zeros of J'_n . Then a spectral sequence for the Neumann problem for $-\Delta_\perp$ on ω is, in addition to the constant eigenfunction,

$$\lambda_{np}^{\text{neu}} = \left(\frac{z'_{np}}{R} \right)^2 \quad \text{and} \quad v_{np}^{\text{neu}} = J_n \left(\frac{z'_{np} r}{R} \right) e^{in\varphi}, \quad n \in \mathbb{Z}, \quad p \geq 1 \quad (7.7)$$

Polarization	k^2	E	H
TE	$(\frac{z'_{np}}{R})^2 + (\frac{m\pi}{\ell})^2$	$\mathbf{M}[v_{np}^{\text{neu}} \otimes \sin(\frac{m\pi}{\ell} \cdot)]$	$\frac{1}{ik} \mathbf{N}[v_{np}^{\text{neu}} \otimes \sin(\frac{m\pi}{\ell} \cdot)]$
TM	$(\frac{z_{np}}{R})^2 + (\frac{m\pi}{\ell})^2$	$\mathbf{N}[v_{np}^{\text{dir}} \otimes \cos(\frac{m\pi}{\ell} \cdot)]$	$ik \mathbf{M}[v_{np}^{\text{dir}} \otimes \cos(\frac{m\pi}{\ell} \cdot)]$

Table 2. Maxwell eigenmodes in a cylinder of radius R and length ℓ , using $\mathbf{M}$ (7.1) – $\mathbf{N}$ (7.2), and v_{np}^{dir} (7.6) – v_{np}^{neu} (7.7).

We summarize results in table 2.

We give in the next table 3 values for the first three zeros $z_{n,j}$ and $z'_{n,j}$ for $n = 0, 1, 2$. We use the relation $J_{\nu-1} - J_{\nu+1} = 2J'_\nu$ to compute $z'_{n,j}$. Since $J_{-1} = -J_1$, we note that there holds

$$z'_{0,j} = z_{1,j}, \quad \forall j \geq 1.$$

$z_{0,j}$	$z_{1,j}$	$z_{2,j}$	$z'_{0,j}$	$z'_{1,j}$	$z'_{2,j}$
2.4048	3.8317	5.1356	3.8317	1.8412	3.0542
5.5201	7.0156	8.4172	7.0156	5.3314	6.7061
8.6537	10.173	11.620	10.173	8.5363	9.9695

Table 3. The first three zeros of $J_0, J_1, J_2, J'_0, J'_1, J'_2$.

7.3 The coaxial cylindrical hole (ω is an annulus)

The boundary conditions on h^{dir} and h^{neu} are now the standard ones at r_0 and R . We have to find the associated eigenpairs of the operator (7.4) for any $n \in \mathbb{N}$. We find that the radial eigenvectors h^{dir} and h^{neu} are linear combinations of the Bessel functions J_n and Y_n of first and second kind:

$$h_{np}^{\text{dir}}(r) = \alpha_{np} J_n(k_{np} r) + \beta_{np} Y_n(k_{np} r)$$

with eigenvalues $\lambda_{np}^{\text{dir}} = (k_{np})^2$, where k_{np} are the positive zeros of the determinant function

$$k \longmapsto J_n(kr_0) Y_n(kR) - Y_n(kr_0) J_n(kR).$$

Analogous formulas exist for h^{neu} .

Since ω has one hole, the number L of the connected components of its boundary is 2. There exists a non-constant harmonic potential v^{top} that takes two distinct constant values on the two connected components of $\partial\omega$. This generator v^{top} can be defined as

$$v^{\text{top}}(x_\perp) = \log r.$$

In connection with Remark 3.4, we note that the conjugate potential $\tilde{v}^{\text{top}}$ is the function $x_{\perp} \mapsto \varphi$. In cylindrical components, there holds

$$\begin{pmatrix} \widetilde{\text{curl}}_{\perp} \tilde{v}^{\text{top}} \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla_{\perp} v^{\text{top}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \text{curl}_{\perp} v^{\text{top}} \\ 0 \end{pmatrix} = - \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \end{pmatrix}.$$

We summarize the results concerning TEM modes for $\Omega = \omega \times I$ with the annulus ω :

Corollary 7.2. *Let ℓ be the length of the cylinder Ω with coaxial hole. Its family of TEM modes is axisymmetric and has the form $(\frac{m\pi}{\ell}, \mathbf{E}^{\text{TEM}}, \mathbf{H}^{\text{TEM}})$ with*

$$\begin{cases} E_r^{\text{TEM}} = \frac{1}{r} \sin(\frac{m\pi}{\ell} x_3), \\ E_{\varphi}^{\text{TEM}} = 0, \\ E_3^{\text{TEM}} = 0, \end{cases} \quad \text{and} \quad \begin{cases} H_r^{\text{TEM}} = 0, \\ H_{\varphi}^{\text{TEM}} = -i \frac{m\pi}{\ell} \frac{1}{r} \cos(\frac{m\pi x_3}{\ell}), \\ H_3^{\text{TEM}} = 0 \end{cases} \quad (7.8)$$

For $k = 0$, the TEM mode is $(k, \mathbf{E}, \mathbf{H}) = (0, \mathbf{0}, \mathbf{H})$ with $\mathbf{H} = (0 \ 1 \ 0)^{\top}$.

Remark 7.3. As r_0 tends to 0, the Dirichlet and Neumann eigenmodes of the annulus tend to the Dirichlet and Neumann eigenvalues of the disc of same radius R . Hence the TE and TM modes of the cylinder with hole tend to the TE and TM modes of the cylinder without hole. In contrast, the TEM modes do not depend on r_0 as long as $r_0 \neq 0$, but disappear at the limit when $r_0 = 0$. This fact has a practical importance when thin conductor wires are present.

8 Maxwell eigenmodes in a ball

For the sake of comparison, we revisit known results about Maxwell eigenmodes in a ball, see [7, Chapter 10]. Let $\Omega \subset \mathbb{R}^3$ be the ball of center 0 and radius R . Here we use spherical coordinates $(\theta, \varphi, \rho) \in [0, \pi] \times \mathbb{T} \times [0, R]$, associate with the orthonormal basis

$$(\hat{\theta}, \hat{\varphi}, \hat{\rho}).$$

Formulas for Maxwell eigenmodes are based on Debye potentials. This is the $\mathbf{M}, \mathbf{N}$ ansatz, in a form slightly different from (2.6): The piloting vector is replaced by the unit field

$$\hat{x} = \frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{i.e.} \quad \hat{x} = \hat{\rho}.$$

The $\mathbf{M}, \mathbf{N}$ ansatz takes the form

$$\mathbf{M}[q] = \text{curl}(q \hat{x}) \quad \text{and} \quad \mathbf{N}[q] = \text{curl} \mathbf{M}[q] = \text{curl} \text{curl}(q \hat{x}). \quad (8.1)$$

Using for instance identities, cf [5, §6.2],

$$\operatorname{curl}(p\mathbf{x}) = \nabla p \times \mathbf{x} \quad \text{and} \quad \operatorname{curl}(\mathbf{a} \times \mathbf{x}) = (\rho \partial_\rho + 2)\mathbf{a} - \mathbf{x} \operatorname{div} \mathbf{a}$$

we find the following formulas where we express vectors in spherical components on the basis $(\hat{\theta}, \hat{\varphi}, \hat{\rho})$

$$\mathbf{M}[q] = \nabla \left(\frac{q}{\rho} \right) \times \mathbf{x} = \nabla q \times \hat{\mathbf{x}} = \begin{pmatrix} \frac{1}{\rho} \partial_\theta q \\ -\frac{1}{\rho \sin \theta} \partial_\varphi q \\ 0 \end{pmatrix} \quad (8.2)$$

and

$$\mathbf{N}[q] = \operatorname{curl} \mathbf{M}[q] = \nabla(\partial_\rho q) - \hat{\mathbf{x}} \rho \Delta \frac{q}{\rho}. \quad (8.3)$$

Therefore

$$\operatorname{curl} \mathbf{N}[q] = \operatorname{curl} \operatorname{curl} \mathbf{M}[q] = -\mathbf{M} \left[\rho \Delta \frac{q}{\rho} \right]. \quad (8.4)$$

Introduce the operator

$$\mathcal{L} : q \longmapsto \mathcal{L}q = -\rho \Delta \frac{q}{\rho}.$$

Then the equations $\operatorname{curl} \operatorname{curl} \mathbf{u} - k^2 \mathbf{u} = 0$ for $\mathbf{M}[q]$ and $\mathbf{N}[q]$ are equivalent to

$$\mathbf{M}[(\mathcal{L} - k^2)q] = 0 \quad \text{and} \quad \mathbf{N}[(\mathcal{L} - k^2)q] = 0.$$

Thus we are interested in scalar solutions q of the equation

$$\mathcal{L}q = k^2 q \quad \text{in} \quad [0, \pi] \times \mathbb{T} \times [0, R]. \quad (8.5)$$

We note that

$$\mathcal{L} = -\partial_\rho^2 - \frac{1}{\rho^2} \Delta_{\mathbb{S}^2} \quad (8.6)$$

with the Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ on the unit sphere $\mathbb{S}^2$

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2.$$

The equation (8.5) is satisfied by all functions in tensor form

$$q(\theta, \varphi, \rho) = Y_n^m(\theta, \varphi) h(k\rho),$$

where Y_n^m are the spherical harmonics and h is a linear combination of the Riccati-Bessel functions ψ_n and χ_n (sometimes written as S_n and C_n). Following Debye's notation, we use the definition

$$\psi_n(x) = x j_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+\frac{1}{2}}(x) \quad \text{and} \quad \chi_n(x) = -x y_n(x) = -\sqrt{\frac{\pi x}{2}} Y_{n+\frac{1}{2}}(x),$$

where J_ν , Y_ν are the Bessel functions and j_n , y_n the spherical Bessel functions of first, second kind, respectively. Because of integrability conditions in 0, χ_n has to be discarded. It remains to look for potentials q of type $Y_n^m \otimes \psi_n(k \cdot)$ so that either $\mathbf{M}[q]$ or $\mathbf{N}[q]$ satisfy the electric boundary condition on the boundary of the ball, i.e.

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{if} \quad \rho = R.$$

Using formulas (8.2) and (8.3), we find that this boundary condition is satisfied by $\mathbf{M}[Y_n^m \otimes \psi_n(k \cdot)]$ if $\psi_n(kR) = 0$ and by $\mathbf{N}[Y_n^m \otimes \psi_n(k \cdot)]$ if $\psi'_n(kR) = 0$. The remarkable fact is that the related potentials are then eigenvectors of the operator $\mathfrak{L}$ with the eigenvalue k^2 for Dirichlet or Neumann conditions. Note that the operator $\mathfrak{L}$ is associated with the coercive bilinear form

$$a(q, \tilde{q}) = \int_0^R \left[\int_{\mathbb{S}^2} \left(\partial_\rho q \partial_\rho \tilde{q} + \frac{1}{\rho^2} \partial_\theta q \partial_\theta \tilde{q} + \frac{1}{\rho^2 \sin^2 \theta} \partial_\varphi q \partial_\varphi \tilde{q} \right) \sin \theta \, d\theta \, d\varphi \right] d\rho$$

on the space

$$V = \{q \in L^2(\mathbb{S}^2 \times (0, R)), \quad \partial_\rho q, \frac{1}{\rho} \nabla_\perp q \in L^2(\mathbb{S}^2 \times (0, R))\}.$$

Completed with either Dirichlet or Neumann boundary conditions on $\rho = R$, $\mathfrak{L}$ is selfadjoint. We have obtained:

Theorem 8.1. (i) *The Dirichlet eigenpairs of $\mathfrak{L}$ have the form $(k_{np}^2, q_{nmp}^{\text{dir}})$, $n \geq 0$, $|m| \leq n$, $p \geq 1$ with $(k_{np})_{p \geq 1}$ the enumeration of the positive zeros of the function $k \mapsto \psi_n(kR)$ and $q_{nmp} = Y_n^m \otimes \psi_n(k_{np} \cdot)$. All triples $(k_{np}, \mathbf{M}[q_{nmp}^{\text{dir}}], \frac{1}{ik} \mathbf{N}[q_{nmp}^{\text{dir}}])$ are Maxwell eigenmodes on the ball of radius R .*

(ii) *The non-constant Neumann eigenpairs of $\mathfrak{L}$ have the form $((k'_{np})^2, q_{nmp}^{\text{neu}})$, $n \geq 0$, $|m| \leq n$, $p \geq 1$ with $(k'_{np})_{p \geq 1}$ the enumeration of the positive zeros of the function $k \mapsto \psi'_n(kR)$ and $q_{nmp} = Y_n^m \otimes \psi_n(k'_{np} \cdot)$. All triples $(k'_{np}, \mathbf{N}[q_{nmp}^{\text{neu}}], ik \mathbf{M}[q_{nmp}^{\text{neu}}])$ are Maxwell eigenmodes on the ball of radius R .*

Remark 8.2. In the literature, the $\mathbf{M}$ ansatz is frequently written in a slightly different way which we distinguish with an asterisk:

$$\mathbf{M}^*[q^*] = \text{curl}(q^* \mathbf{x})$$

instead of $\mathbf{M}[q] = \text{curl}(q \hat{\mathbf{x}})$. As usual, $\mathbf{N}^* = \text{curl} \mathbf{M}^*$. The outcome for the Maxwell eigenmodes is the same of course. Nevertheless, the interpretation of the potentials is different. We have

$$q^* = \frac{q}{\rho}.$$

- (i) Concerning Dirichlet modes, the functions q_{nmp}^* defined as $q_{nmp}^{\text{dir}}/\rho$ are the eigenfunctions of the Dirichlet problem for the standard positive Laplace operator $-\Delta$ on the ball. In other words, the eigenvalues k_{np}^2 are also the standard Laplace eigenvalues.

- (ii) But, when Neumann modes are concerned, the functions q_{nmp}^* defined as $q_{nmp}^{\text{neu}}/\rho$ are not Neumann eigenvalues for $-\Delta$.

Remark 8.3. The tensor product potentials $Y_n^m \otimes h(k \cdot)$ with h being any of the Riccati-Bessel functions have been used more than a century ago to describe scattering of plane waves by a dielectric sphere (Mie series). Scattering resonances (with negative imaginary part) have also been investigated at that time. More recently, whispering gallery modes have been analytically calculated by a similar method [2]. All these problems are transmission problems between the ball and its exterior. Inside the ball h has the form $\psi_n(n_{\text{opt}}k \cdot)$ where n_{opt} is the refractive (or optical) index of the ball. Outside the ball, h is either $\zeta_n^{(1)}(k \cdot)$ for scattering, or $\chi_n(k \cdot)$ for whispering gallery modes.

9 Extension to nonconstant electric permittivity

Let us consider the original Maxwell system (A.4). We still assume that the magnetic permeability μ is equal to μ_0 in the whole domain Ω . But we allow now that the electric permittivity ε may vary in Ω . We set

$$\varepsilon = \varepsilon_{\text{rel}}\varepsilon_0, \quad \varepsilon_{\text{rel}} \geq 1.$$

We consider domains Ω in the product form $\omega \times I$. We assume that

$$\varepsilon_{\text{rel}}(x) = \varepsilon_{\text{rel}}(x_{\perp}), \quad \varepsilon_{\text{rel}} \in L^{\infty}(\omega), \quad (9.1)$$

like in wave guides or optic fibers. The Maxwell system takes now the form (A.6) instead of (1.1). Then the classification of eigenvectors into TE, TM and TEM does not hold any more (at least not in the form given by Theorem 3.6 and Corollary 4.1). Nevertheless, the splitting of the spectrum according to frequencies with respect to the axial variable x_3 remains possible, as well as a tensor product form. We are going to investigate the magnetic field $\mathbf{H}$, taking advantage of its local regularity even if ε_{rel} is not continuous. The magnetic variational formulation becomes, instead of (2.2):

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ with $\mathbf{u} \neq 0$ in $\mathbf{X}_{\text{T}}(\Omega)$ and $\text{div } \mathbf{u} = 0$ such that

$$\int_{\Omega} \frac{1}{\varepsilon_{\text{rel}}} \text{curl } \mathbf{u} \text{ curl } \mathbf{u}' + s \text{div } \mathbf{u} \text{ div } \mathbf{v} \, \text{d}\mathbf{x} = \Lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{u}' \, \text{d}\mathbf{x}, \quad \forall \mathbf{u}' \in \mathbf{X}_{\text{T}}(\Omega), \quad (9.2)$$

Here s is nonnegative. The choice $s > 0$ corresponds to an elliptic regularization of the system. To simplify notations, let us assume that

$$\boxed{I = (0, \pi)} \quad (9.3)$$

In the constant material case, considering the Maxwell eigenmodes from the magnetic point of view, we note that the magnetic part of eigenmodes given in Corollary 4.1

have the following form

$$\mathbf{H}^{\text{TE}} = \begin{pmatrix} k \nabla_{\perp} v(x_{\perp}) \cos(mx_3) \\ -\Delta_{\perp} v(x_{\perp}) \sin(mx_3) \end{pmatrix} \quad \text{and} \quad \mathbf{H}^{\text{TM}} = \begin{pmatrix} \text{curl}_{\perp} v(x_{\perp}) \cos(mx_3) \\ 0 \end{pmatrix} \quad (9.4)$$

We are going to prove that we still have a similar structure with respect to the axial variable x_3 .

Theorem 9.1. *With the assumptions (9.1) and (9.3), the magnetic eigenmodes solution of (9.2) can be organized in a sequence of independent families $\mathfrak{H}_m$ with index $m \in \mathbb{N}$ in which each eigenvector has the tensor product form*

$$\mathbf{H} = \begin{pmatrix} \mathbf{v}_{\perp}(x_{\perp}) \cos(mx_3) \\ v_3(x_{\perp}) \sin(mx_3) \end{pmatrix}. \quad (9.5)$$

For any $m \in \mathbb{N}$, let Λ_j^m and $\mathbf{v}_j^m := (\mathbf{v}_{\perp,j}^m, v_{3,j}^m)$ be the eigenpairs of the problem:

Find $\Lambda \in \mathbb{R}$, $\mathbf{v} = (\mathbf{v}_{\perp}, v_3) \neq 0$ in $\mathbf{X}_{\text{T}}(\omega) \times H^1(\omega)$ with $\text{div}_{\perp} \mathbf{v}_{\perp} + m v_3 = 0$ such that

$$\begin{aligned} \int_{\omega} \frac{1}{\varepsilon_{\text{rel}}} \left\{ \text{curl}_{\perp} \mathbf{v}_{\perp} \text{curl}_{\perp} \mathbf{v}'_{\perp} + (\nabla_{\perp} v_3 + m \mathbf{v}_{\perp}) \cdot (\nabla_{\perp} v'_3 + m \mathbf{v}'_{\perp}) \right\} \mathbf{d}\mathbf{x} \\ = \Lambda \int_{\omega} \mathbf{v} \cdot \mathbf{v}' \mathbf{d}\mathbf{x}, \quad \forall \mathbf{v}' \in \mathbf{X}_{\text{T}}(\omega) \times H^1(\omega). \end{aligned} \quad (9.6)$$

Denote by $\mathbf{H}_j^m$ the vector of form (9.5) with $\mathbf{v} = \mathbf{v}_j^m$. Then the eigenpairs $(\Lambda_j^m, \mathbf{H}_j^m)_{j \geq 1}$ span the family $\mathfrak{H}_m$.

Remark 9.2. The bilinear form of problem (9.6) can be regularized by

$$\int_{\omega} \frac{1}{\varepsilon_{\text{rel}}} \left\{ (\text{div}_{\perp} \mathbf{v}_{\perp} + m v_3) (\text{div}_{\perp} \mathbf{v}'_{\perp} + m v'_3) \right\} \mathbf{d}\mathbf{x}.$$

We can check that if ε_{rel} is constant, the resulting bilinear form is equal to

$$\begin{aligned} \frac{1}{\varepsilon_{\text{rel}}} \int_{\omega} \text{curl}_{\perp} \mathbf{v}_{\perp} \text{curl}_{\perp} \mathbf{v}'_{\perp} + \nabla_{\perp} v_3 \cdot \nabla_{\perp} v'_3 \\ + \text{div}_{\perp} \mathbf{v}_{\perp} \text{div}_{\perp} \mathbf{v}'_{\perp} + m^2 (\mathbf{v}_{\perp} \cdot \mathbf{v}'_{\perp} + v_3 v'_3) \mathbf{d}\mathbf{x}. \end{aligned}$$

Proof of Theorem 9.1. Solutions of (9.2) satisfy on $\omega \times \{0\}$ the essential boundary condition $u_3 = 0$, and the natural boundary condition $\frac{1}{\varepsilon_{\text{rel}}} \text{curl} \mathbf{u} \times \mathbf{e}_3 = 0$. Since $u_3 = 0$ on $\omega \times \{0\}$, $\partial_1 u_3$ and $\partial_2 u_3$ are also 0 on $\omega \times \{0\}$, and the natural boundary condition implies that $\partial_3 u_1 = \partial_3 u_2 = 0$ on $\omega \times \{0\}$. Therefore, defining the extension

$$\tilde{\mathbf{u}}_{\perp}(x_{\perp}, -x_3) = \mathbf{u}_{\perp}(x_{\perp}, x_3) \quad \text{and} \quad \tilde{u}_3(x_{\perp}, -x_3) = -u_3(x_{\perp}, x_3), \quad \forall x_3 \in (0, \pi)$$

we obtain an element $\tilde{\mathbf{u}} \in \mathbf{X}_T(\omega \times (-\pi, \pi))$ which satisfies $\operatorname{div} \tilde{\mathbf{u}} = 0$ and is solution of (9.2) on the extended domain $\omega \times (-\pi, \pi)$. Moreover, $\mathbf{u}(x_\perp, -\pi) = \mathbf{u}(x_\perp, \pi)$ and $\partial_3 \mathbf{u}(x_\perp, -\pi) = \partial_3 \mathbf{u}(x_\perp, \pi)$ for all $x_\perp \in \omega$. We deduce that $\tilde{\mathbf{u}}$ is solution of (9.2) on the domain $\mathbf{X}_T(\omega \times \mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Since the coefficient ε_{rel} does not depend on x_3 , the underlying Maxwell operator commutes with ∂_3 . Therefore the spectrum of problem (9.2) can be decomposed according to the eigenvectors of ∂_3 on $\mathbb{T}$, which are the functions $x_3 \mapsto e^{imx_3}$, $m \in \mathbb{Z}$.

For any positive integer m , we notice that if $(\mathbf{v}_\perp(x_\perp), v_3(x_\perp))e^{imx_3}$ is solution of (9.2) on the domain $\mathbf{X}_T(\omega \times \mathbb{T})$, then $(\mathbf{v}_\perp(x_\perp), -v_3(x_\perp))e^{-imx_3}$ is also solution of the same problem. Therefore, their sum is also solution of the same problem. Moreover this sum has the form (9.5) and satisfies the boundary conditions (perfectly conducting walls)¹ of the space $\mathbf{X}_T(\Omega)$. Conversely this sum is, up to a multiplicative constant, the only linear combination of $(\mathbf{v}_\perp(x_\perp), v_3(x_\perp))e^{imx_3}$ and $(\mathbf{v}_\perp(x_\perp), -v_3(x_\perp))e^{-imx_3}$ which satisfies the boundary conditions of the space $\mathbf{X}_T(\Omega)$.

Calculating

$$\int_{\Omega} \frac{1}{\varepsilon_{\text{rel}}} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{u}' \, d\mathbf{x}$$

for

$$\mathbf{u} = \begin{pmatrix} \mathbf{v}(x_\perp) \cos(mx_3) \\ v_3(x_\perp) \sin(mx_3) \end{pmatrix} \quad \text{and} \quad \mathbf{u}' = \begin{pmatrix} \mathbf{v}'(x_\perp) \cos(mx_3) \\ v'_3(x_\perp) \sin(mx_3) \end{pmatrix},$$

we find

$$\int_{\omega} \frac{1}{\varepsilon_{\text{rel}}} \left\{ \operatorname{curl}_\perp \mathbf{v}_\perp \operatorname{curl}_\perp \mathbf{v}'_\perp + (\operatorname{curl}_\perp v_3 + m\mathbf{v}_\perp \times \mathbf{e}_3) \cdot (\operatorname{curl}_\perp v'_3 + m\mathbf{v}'_\perp \times \mathbf{e}_3) \right\} d\mathbf{x}$$

which coincides with the bilinear form in problem (9.6) □

Remark 9.3. For $m = 0$, problem (9.6) reduces to two uncoupled problems: The magnetic 2D Maxwell eigenvalue problem in ω for $\mathbf{v}_\perp$ and the Neumann eigenvalue problem for $-\Delta_\perp$ in ω for v_3 . This last problem does not yield any non-trivial solution of (9.6) since for $m = 0$, the third component in the Ansatz (9.5) is zero. Moreover, we can show that the solutions of the magnetic 2D Maxwell eigenvalue problem in ω are the pairs $(\operatorname{curl}_\perp v_j^{\text{dir}}, \lambda_j^{\text{dir}})$, $j \geq 1$, with the eigenpairs $(v_j^{\text{dir}}, \lambda_j^{\text{dir}})$ of the problem

$$-\Delta_\perp v = \lambda \varepsilon v \quad \text{in } \omega, \quad v \in H_0^1(\omega). \quad (9.7)$$

Thus we have found for $m = 0$ the family of TM modes:

$$\mathbf{H}_j^{\text{TM}} = \begin{pmatrix} \operatorname{curl}_\perp v_j^{\text{dir}}(x_\perp) \\ 0 \end{pmatrix} \quad j \geq 1.$$

¹ Considering the *difference* instead the sum, we would find the perfectly insulating boundary conditions on $\omega \times \partial I$.

A Normalizing Maxwell equations

Let ε and μ are the electric permittivity and the magnetic permeability of the material inside Ω . We assume that the boundary of Ω represents *perfectly conducting* or *perfectly insulating* walls:

$$\partial\Omega = \partial\Omega_{\text{cd}} \cup \partial\Omega_{\text{ins}}, \quad \partial\Omega_{\text{cd}} \cap \partial\Omega_{\text{ins}} = \emptyset, \quad (\text{A.1})$$

where $\partial_{\text{cd}}\Omega$ is the perfectly conducting part and $\partial_{\text{ins}}\Omega$ the perfectly insulating part.

The cavity resonator problem is to find the frequencies $\varpi \in \mathbb{R}_+$ and the non-zero electromagnetic fields $(\hat{\mathbf{E}}, \hat{\mathbf{H}}) \in L^2(\Omega)^6$ such that

$$\begin{cases} \text{curl } \hat{\mathbf{E}} - i\varpi\mu\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad (\text{Faraday law}) \\ \text{curl } \hat{\mathbf{H}} + i\varpi\varepsilon\hat{\mathbf{E}} = 0 & \text{in } \Omega, \quad (\text{Ampère law}) \\ \text{div } \varepsilon\hat{\mathbf{E}} = 0 \quad \text{and} \quad \text{div } \mu\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad (\text{gauge conditions}). \end{cases} \quad (\text{A.2a})$$

with boundary conditions

$$\begin{cases} \hat{\mathbf{E}} \times \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \cdot \mathbf{n} = 0, & \text{on } \partial\Omega_{\text{cd}}, \quad (\text{perfect conductor b. c.}) \\ \hat{\mathbf{E}} \cdot \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \times \mathbf{n} = 0, & \text{on } \partial\Omega_{\text{ins}}, \quad (\text{perfect insulator b. c.}) \end{cases} \quad (\text{A.2b})$$

In this paper, we consider the non-magnetic case, i.e. when $\mu \equiv \mu_0$ in Ω . We can set

$$\varepsilon = n_{\text{opt}}^2 \varepsilon_0 = \varepsilon_{\text{rel}} \varepsilon_0 \quad (\text{A.3})$$

where n_{opt} is the refractive index of the material and ε_{rel} the relative permittivity. Then (A.2a) reduces to

$$\begin{cases} \text{curl } \hat{\mathbf{E}} - i\varpi\mu_0\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad (\text{Faraday law}) \\ \text{curl } \hat{\mathbf{H}} + i\varpi\varepsilon_{\text{rel}}\varepsilon_0\hat{\mathbf{E}} = 0 & \text{in } \Omega, \quad (\text{Ampère law}) \\ \text{div } \varepsilon_{\text{rel}}\varepsilon_0\hat{\mathbf{E}} = 0 \quad \text{and} \quad \text{div } \mu_0\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad (\text{gauge conditions}). \end{cases} \quad (\text{A.4})$$

With the normalization

$$k = \varpi\sqrt{\varepsilon_0\mu_0} \quad (\text{wave number}), \quad \mathbf{E} = \sqrt{\varepsilon_0}\hat{\mathbf{E}} \quad \text{and} \quad \mathbf{H} = \sqrt{\mu_0}\hat{\mathbf{H}}. \quad (\text{A.5})$$

system (A.4) is transformed into

$$\begin{cases} \text{curl } \mathbf{E} - ik\mathbf{H} = 0 & \text{in } \Omega, \\ \text{curl } \mathbf{H} + ik\varepsilon_{\text{rel}}\mathbf{E} = 0 & \text{in } \Omega, \\ \text{div } \varepsilon_{\text{rel}}\mathbf{E} = 0 \quad \text{and} \quad \text{div } \mathbf{H} = 0 & \text{in } \Omega. \end{cases} \quad (\text{A.6})$$

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