

The Influence of Lateral Boundary Conditions on the Asymptotics in Thin Elastic Plates I: Clamped and Simply Supported Plates

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Abstract. *In this series of two papers, we investigate the limit behaviors and the boundary layers of the three-dimensional displacement in thin elastic plates as the thickness tends to zero, in each of the eight main types of lateral boundary conditions on their edges: hard and soft clamped, hard and soft simple support, friction conditions, sliding edge and free plates. This part is devoted to the first four types of conditions. Relying on construction algorithms [6, 7], we establish an asymptotics of the displacement combining inner and outer expansions. We describe the two first terms in the outer expansion: these are Kirchhoff-Love displacements satisfying prescribed boundary conditions that we exhibit. We also study the first boundary layer term: it has generically non-zero transverse and normal components in each of these four types of lateral conditions.*

INTRODUCTION

The analysis as the thickness tends to zero of characteristic fields and tensors, solutions of the three-dimensional equations of linearized elastostatics in thin elastic plates has a twofold interest: On one hand the singular perturbation nature of the problem is both a mechanical and mathematical challenge, and on the other hand the knowledge of accurate asymptotics allows the construction of simplified and performing numerical models.

In the case when the plate is clamped along its lateral boundary, the situation is now well-known, at least theoretically: The comparison between 3D and 2D models was first performed by the construction of infinite *formal* asymptotic expansions, see FRIEDRICHS & DRESSLER [9], GOL'DENVEIZER [10], GREGORY & WAN [11]. Shortly before, MORGENSTERN [16] was indeed the first to prove that the KIRCHHOFF model [12] is the correct asymptotic limit of the 3D model when the thickness approaches zero in the hard clamped, hard simply supported and free plate situations by using the PRAGER-SYNGE hypercircle theorem [20]. Next, rigorous error estimates between the 3D solution and its limit were proved by SHOIKHET [21] and by CIARLET and DESTUYNDER [4, 8, 3]. Further terms were exhibited by NAZAROV & ZORIN [18], and the whole asymptotic expansion was constructed by DAUGE & GRUAIS [6, 7].

Different types of lateral boundary conditions are of interest in mechanics: let us quote the soft clamped plate where the tangential in-plane component of the displacement is free, the hard simply supported plate where the normal component of the displacement is free, the soft simply supported plate where both above components of the displacement are free, and also the totally free plate. These cases were investigated by ARNOLD & FALK [1] where an asymptotics for the Reissner-Mindlin plate was constructed, and by CHEN [2] where error bounds between the 3D solution and its limit were evaluated.

In this series of two papers, we prove the validity of an infinite asymptotic expansion of the displacement with optimal error estimates in H^1 , L^2 and energy norms. Such an expansion can be differentiated and provides then corresponding results for the stress and the strain tensors, see [5] for the clamped case. Like in [18] and [6, 7], this asymptotics includes

- An outer part containing displacements only depending on the in-plane variables x_* and on the scaled transverse variable x_3 ,
- An inner part containing exponentially decaying profiles (boundary layer terms), depending on two scaled variables (x_3 and $t = r/\varepsilon$ where r is the distance to the lateral boundary).

As material law, we choose to remain in the framework of homogeneous, isotropic materials. Such a property allows to uncouple the boundary layer terms φ into two parts:

- The horizontal tangential component φ_s governed by a Laplace equation,
- The two other components φ_t and φ_3 governed by the bi-dimensional Lamé equations, whose solutions can themselves be uncoupled in membrane and bending modes, i.e. possessing parity properties with respect to the transverse variable: the former having an even φ_t and an odd φ_3 and the latter having converse properties.

Thus, conditions ensuring the exponential decay at infinity of solutions of the above problems can be made explicit, resulting into simple coupling formulas between the inner and outer parts of the expansion. And in the end these coupling formulas lead to the determination of boundary conditions for the limit problems (membrane and bending) showing up as compatibility conditions in the calculation of the outer expansion.

This paper contains eight sections, organized as follows: in section 1 we introduce the problems on the thin plates, the eight sorts of lateral boundary conditions, the local and scaled coordinates, and the scaled form of the problems; after that we have enough material to present an outline of our results in the form of several tables. In section 2 we formulate the mixed Ansatz containing outer and inner expansion parts and give as an algorithm the construction rules for the outer part, while in section 3 we formulate the boundary value problems on a half-strip governing the boundary layer profiles φ and give in section 4 the conditions on the data ensuring the existence of exponentially decreasing solutions to these problems. The four last sections are devoted to each of the four types of lateral boundary conditions : hard and soft clamped, hard and soft simple supported plates.

1 LATERAL BOUNDARY CONDITIONS

We aim to study the behavior of the displacement field $\mathbf{u}^\varepsilon$ in a family of thin elastic three-dimensional plates Ω^ε as the thickness parameter ε tends to zero. The plates Ω^ε are defined as follows:

$$\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon) \quad \text{with } \omega \subset \mathbb{R}^2 \text{ a regular domain and } \varepsilon > 0,$$

and are constituted of a homogeneous, isotropic material with Lamé constants λ and μ .

1.1 The equations of elasticity on thin plates

As standard, let $\mathbf{u} = (u_1, u_2, u_3)$ denote the displacement field, and let $e(\mathbf{u})$ denote the associated linearized strain tensor $e_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Then the stress tensor $\sigma(\mathbf{u})$ is given by Hooke's law

$$\sigma(\mathbf{u}) = A e(\mathbf{u}),$$

where the rigidity matrix $A = (A_{ijkl})$ of the material is given by:

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

The inward traction field at a point on the boundary is $\mathbf{T} = (T_1, T_2, T_3)$ defined as $\sigma(\mathbf{u}) \mathbf{n}$ where $\mathbf{n}$ is the unit interior normal to the boundary.

In the sequel, we make the assumption that the boundary conditions on the upper and lower faces $\Gamma_\pm^\varepsilon := \omega \times \{-\varepsilon, +\varepsilon\}$ are of traction type. On the lateral face

$$\Gamma_0^\varepsilon := \partial\omega \times (-\varepsilon, +\varepsilon),$$

we are going to consider (in this paper and in its part II) the 8 ‘canonical’ choices of boundary conditions which will be denoted by $\textcircled{i}$ where $i = 1, \dots, 8$. Indeed, on the lateral boundary Γ_0^ε we can distinguish three natural components in the displacements or the tractions: normal, horizontal tangential, vertical, and we obtain 8 ‘canonical’ lateral boundary conditions, according to how we choose to prescribe the displacement or the traction for each component.

To $\mathbf{u}$ we associate its normal component $u_n := \mathbf{u} \cdot \mathbf{n} = u_1 n_1 + u_2 n_2$ on Γ_0^ε and its horizontal tangential component $u_s := u_1 n_2 - u_2 n_1$, the vertical component being u_3 and similar notations apply to $\mathbf{T}$. To each boundary condition $\textcircled{i}$ corresponds two complementary sets of indices $A_{\textcircled{i}}$ and $B_{\textcircled{i}}$ where $A_{\textcircled{i}}$ is attached to the Dirichlet conditions of $\textcircled{i}$, i.e. $u_a = 0$ for each index $a \in A_{\textcircled{i}}$: these are the *stable* conditions. The Neumann conditions are $T_b = 0$ for each index $b \in B_{\textcircled{i}}$ and appear as *natural* conditions.

Here is the table of the eight lateral boundary conditions (in boldface, those we treat in this first part).

① Type	Dirichlet	Neumann	$A_{\textcircled{1}}$	$B_{\textcircled{1}}$
① hard clamped	$\mathbf{u} = 0,$		$\{n, s, 3\}$	
② soft clamped	$u_n, u_3 = 0,$	$T_s = 0$	$\{n, 3\}$	$\{s\}$
③ hard simply supported	$u_s, u_3 = 0,$	$T_n = 0$	$\{s, 3\}$	$\{n\}$
④ soft simply supported	$u_3 = 0,$	$T_n, T_s = 0$	$\{3\}$	$\{n, s\}$
⑤ frictional I	$u_n, u_s = 0,$	$T_3 = 0$	$\{n, s\}$	$\{3\}$
⑥ sliding edge	$u_n = 0,$	$T_s, T_3 = 0$	$\{n\}$	$\{s, 3\}$
⑦ frictional II	$u_s = 0,$	$T_n, T_3 = 0$	$\{s\}$	$\{n, 3\}$
⑧ free		$\mathbf{T} = 0$		$\{n, s, 3\}$

Table 1. Lateral boundary conditions.

To each boundary condition ① is associated the space of displacements $V_{\textcircled{1}}(\Omega^\varepsilon)$

$$V_{\textcircled{1}}(\Omega^\varepsilon) = \{\mathbf{v} \in H^1(\Omega^\varepsilon)^3 \mid v_a = 0 \text{ on } \Gamma_0^\varepsilon \quad \forall a \in A_{\textcircled{1}}\}$$

and the space $\mathcal{R}_{\textcircled{1}}$ of the rigid motions satisfying the Dirichlet conditions of $V_{\textcircled{1}}$. Then, the variational formulation of the problem consists in finding

$$\begin{cases} \mathbf{u}^\varepsilon \in V_{\textcircled{1}}(\Omega^\varepsilon) \\ \forall \mathbf{v} \in V_{\textcircled{1}}(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} A e(\mathbf{u}^\varepsilon) : e(\mathbf{v}) = \int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \cdot \mathbf{v} + \int_{\Gamma_+^\varepsilon} \mathbf{g}^{\varepsilon,+} \cdot \mathbf{v} - \int_{\Gamma_-^\varepsilon} \mathbf{g}^{\varepsilon,-} \cdot \mathbf{v}, \end{cases} \quad (1.1)$$

where $\mathbf{f}^\varepsilon$ represents the volume force and $\mathbf{g}^{\varepsilon,\pm}$ the prescribed horizontal tractions. If the right hand side satisfies the compatibility condition

$$\forall \mathbf{v} \in \mathcal{R}_{\textcircled{1}}(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \cdot \mathbf{v} + \int_{\Gamma_+^\varepsilon} \mathbf{g}^{\varepsilon,+} \cdot \mathbf{v} - \int_{\Gamma_-^\varepsilon} \mathbf{g}^{\varepsilon,-} \cdot \mathbf{v} = 0, \quad (1.2)$$

then there exists a unique solution to (1.1) satisfying

$$\forall \mathbf{v} \in \mathcal{R}_{\textcircled{1}}(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} \mathbf{u}^\varepsilon \cdot \mathbf{v} = 0. \quad (1.3)$$

Denoting by the Greek indices α, β, γ the values in $\{1, 2\}$ corresponding to the in-plane variables, we can write the equations inside Ω^ε as

$$(\lambda + \mu) \partial_\alpha \operatorname{div} \mathbf{u}^\varepsilon + \mu \Delta u_\alpha^\varepsilon = -f_\alpha^\varepsilon, \quad \alpha = 1, 2, \quad (1.4a)$$

$$(\lambda + \mu) \partial_3 \operatorname{div} \mathbf{u}^\varepsilon + \mu \Delta u_3^\varepsilon = -f_3^\varepsilon. \quad (1.4b)$$

The boundary conditions on the horizontal sides $\Gamma_\pm^\varepsilon$ are

$$2\mu e_{\alpha 3}(\mathbf{u}^\varepsilon) = g_\alpha^{\varepsilon,\pm}, \quad \alpha = 1, 2, \quad (1.5a)$$

$$2\mu \partial_3 u_3^\varepsilon + \lambda \operatorname{div} \mathbf{u}^\varepsilon = g_3^{\varepsilon,\pm}. \quad (1.5b)$$

The boundary conditions on the lateral side Γ_0^ε can be written as

$$u_a^\varepsilon = 0, \quad \forall a \in A_{\textcircled{1}} \quad \text{and} \quad T_b^\varepsilon = 0, \quad \forall b \in B_{\textcircled{1}}. \quad (1.6)$$

1.2 Local coordinates

In order to obtain precise formulas for our asymptotics, we have to use local coordinates (r, s) in a neighborhood of the lateral boundary Γ_0^ε . Of course, it is enough to define such local coordinates in a plane neighborhood of $\partial\omega$.

Let $\mathbf{n}$ be the inner unit normal to $\partial\omega$ and $\boldsymbol{\tau}$ be the tangent unit vector field to $\partial\omega$ such that the basis $(\boldsymbol{\tau}, \mathbf{n})$ is direct in each point of $\partial\omega$. Denote by s a curvilinear abscissa (arc length) along $\partial\omega$ oriented according to $\boldsymbol{\tau}$. Let $\mathbb{S} \sim \partial\omega$ be the set of the values of s :

$$\mathbb{S} \ni s \longmapsto \boldsymbol{\gamma}(s) \in \partial\omega.$$

For a point $x_* \in \mathbb{R}^2$, let $r = r(x_*)$ be its signed distance to $\partial\omega$ oriented along $\mathbf{n}$, i.e. r is this distance if $x_* \in \omega$ and r is minus this distance if $x_* \notin \omega$. If $|r|$ is small enough, there exists a unique point $x_*^0 \in \partial\omega$ such that $|r| = \text{dist}(x_*, x_*^0)$ and we define $s = s(x_*)$ as the curvilinear abscissa of x_*^0 . Thus, we have a tubular neighborhood of $\partial\omega$ which is diffeomorphic to $(-r^0, r^0) \times \mathbb{S}$ via the change of variables $x_* \mapsto (r, s)$. And, in this tubular neighborhood, the partial derivatives ∂_r and ∂_s are well defined (and, of course, commute with each other).

We extend the vector fields $\mathbf{n}$ and $\boldsymbol{\tau}$ from $\mathbb{S}$ to $(-r^0, r^0) \times \mathbb{S}$ by

$$\forall r \in (-r^0, r^0), \quad \forall s \in \mathbb{S}, \quad \mathbf{n}(r, s) = \mathbf{n}(s) \quad \text{and} \quad \boldsymbol{\tau}(r, s) = \boldsymbol{\tau}(s).$$

We have

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau} = \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix}.$$

Moreover, we have the identities

$$\partial_r \mathbf{n} = 0 \quad \text{and} \quad \partial_r \boldsymbol{\tau} = 0$$

and Frenet's relations

$$\partial_s \mathbf{n} = -\frac{1}{R} \boldsymbol{\tau} \quad \text{and} \quad \partial_s \boldsymbol{\tau} = \frac{1}{R} \mathbf{n},$$

where $R = R(s)$ is the curvature radius of $\partial\omega$ at s from inside ω . Let us recall that $\frac{1}{R} = \kappa$ is the curvature. Thus, relying on the relation $x_* = \boldsymbol{\gamma}(s) + r \mathbf{n}(s)$, we obtain

$$\partial_r = n_1 \partial_1 + n_2 \partial_2 \quad \text{and} \quad \partial_s = \left(1 - \frac{r}{R}\right)(n_2 \partial_1 - n_1 \partial_2). \quad (1.7)$$

Of course $\partial_n = \partial_r$. In view of (1.7) we introduce the following extension to $(-r^0, r^0) \times \mathbb{S}$ of the functions u_n, u_s, T_n, T_s which were formerly defined on $\partial\omega$:

$$u_n = n_1 u_1 + n_2 u_2 \quad \text{and} \quad u_s = \left(1 - \frac{r}{R}\right)(n_2 u_1 - n_1 u_2), \quad (1.8a)$$

$$T_n = n_1 T_1 + n_2 T_2 \quad \text{and} \quad T_s = \left(1 - \frac{r}{R}\right)(n_2 T_1 - n_1 T_2). \quad (1.8b)$$

With these definitions, we obtain that the normal, tangential horizontal and (tangential) vertical components of the traction $\mathbf{T}^\varepsilon$ on Γ_0^ε are given by respectively

$$T_n^\varepsilon = \lambda \operatorname{div} \mathbf{u}^\varepsilon + 2\mu \partial_n u_n^\varepsilon, \quad (1.9a)$$

$$T_s^\varepsilon = \mu(\partial_s u_n^\varepsilon + \partial_n u_s^\varepsilon + \frac{2}{R} u_s^\varepsilon), \quad (1.9b)$$

$$T_3^\varepsilon = \mu(\partial_n u_3^\varepsilon + \partial_3 u_n^\varepsilon). \quad (1.9c)$$

1.3 The scaling

To study the behavior of the unknown $\mathbf{u}^\varepsilon$ as ε tends to zero, we make a dilatation along the vertical axis ($x_3 = \varepsilon^{-1} \tilde{x}_3$) in order to have a fixed reference configuration Ω :

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \Omega^\varepsilon \mapsto x = (x_1, x_2, x_3) \in \Omega, \quad \text{where } \Omega = \omega \times (-1, +1). \quad (1.10)$$

Then, the new unknown displacement is a vector field denoted by $\mathbf{u}(\varepsilon)$ which we define for purpose of homogeneity by:

$$u_\alpha(\varepsilon)(x) = u_\alpha^\varepsilon(\tilde{x}), \quad \alpha = 1, 2, \quad u_3(\varepsilon)(x) = \varepsilon u_3^\varepsilon(\tilde{x}). \quad (1.11)$$

We may assume without theoretical restriction that the asymptotic expansion of the unknown displacement will begin with a zero-th order term in ε : to this aim, we make an additional assumption on the right hand side, namely

$$f_\alpha^\varepsilon(\tilde{x}) = f_\alpha(x), \quad \alpha = 1, 2, \quad \varepsilon^{-1} f_3^\varepsilon(\tilde{x}) = f_3(x), \quad (1.12a)$$

$$\varepsilon^{-1} g_\alpha^{\varepsilon, \pm}(\tilde{x}) = g_\alpha^\pm(x_*), \quad \alpha = 1, 2, \quad \varepsilon^{-2} g_3^{\varepsilon, \pm}(\tilde{x}) = g_3^\pm(x_*). \quad (1.12b)$$

To simplify, we assume that the data $\mathbf{f}$ and $\mathbf{g}^\pm$ are regular up to the boundary, i.e. $\mathbf{f} \in \mathcal{C}^\infty(\overline{\Omega})^3$ and $\mathbf{g}^\pm \in \mathcal{C}^\infty(\overline{\omega})^3$. This suitable scaling of the data is proved as necessary in MIARA [14], where it is only assumed that the loads can be given arbitrarily with respect to x (and of course that the expansion starts with ε^0). So this assumption of scaling is the correct one.

After the scaling (1.11), (1.12), problem (1.1) is transformed into the new boundary value problem (1.13)-(1.15) on Ω , whose solution is $\mathbf{u} = \mathbf{u}(\varepsilon)$. To formulate it, we use the repeated index convention. Moreover $\mathbf{u}_*$ is a condensed notation for (u_1, u_2) , $\operatorname{div}_* \mathbf{u}_*$ denotes $\partial_1 u_1 + \partial_2 u_2$ and Δ_* denotes the horizontal Laplacian $\partial_{11} + \partial_{22}$.

The in-plane components, indexed by $\alpha = 1, 2$, and the vertical component of the interior equations in Ω are:

$$2\mu \partial_3 e_{\alpha 3}(\mathbf{u}) + \lambda \partial_{\alpha 3} u_3 + \varepsilon^2 \left((\lambda + \mu) \partial_\alpha \operatorname{div}_* \mathbf{u}_* + \mu \Delta_* u_\alpha \right) = -\varepsilon^2 f_\alpha, \quad (1.13a)$$

$$(\lambda + 2\mu) \partial_{33} u_3 + \varepsilon^2 \left(\lambda \partial_3 \operatorname{div}_* \mathbf{u}_* + 2\mu \partial_\gamma e_{\gamma 3}(\mathbf{u}) \right) = -\varepsilon^4 f_3. \quad (1.13b)$$

The boundary conditions on the horizontal sides $\Gamma_{\pm} := \{x_3 = \pm 1\} \cap \partial\Omega$ are

$$2\mu e_{\alpha 3}(\mathbf{u}) = \varepsilon^2 g_{\alpha}^{\pm}, \quad \alpha = 1, 2, \quad (1.14a)$$

$$(\lambda + 2\mu)\partial_3 u_3 + \varepsilon^2 \lambda \operatorname{div}_* \mathbf{u}_* = \varepsilon^4 g_3^{\pm}. \quad (1.14b)$$

The boundary conditions on the lateral side $\Gamma_0 = \partial\omega \times (-1, 1)$ can be written as

$$u_a = 0, \quad \forall a \in A_{\textcircled{1}} \quad \text{and} \quad T_b = 0, \quad \forall b \in B_{\textcircled{1}}. \quad (1.15)$$

The normal, tangential horizontal and vertical components of the traction $\mathbf{T} = \mathbf{T}(\varepsilon)$ on Γ_0 are given by respectively

$$T_n(\varepsilon) = \lambda \partial_3 u_3(\varepsilon) + \varepsilon^2 (\lambda \operatorname{div}_* \mathbf{u}_*(\varepsilon) + 2\mu \partial_n u_n(\varepsilon)), \quad (1.16a)$$

$$T_s(\varepsilon) = \varepsilon^2 \mu (\partial_s u_n(\varepsilon) + \partial_n u_s(\varepsilon) + \frac{2}{R} u_s(\varepsilon)), \quad (1.16b)$$

$$T_3(\varepsilon) = \mu (\partial_n u_3(\varepsilon) + \partial_3 u_n(\varepsilon)). \quad (1.16c)$$

1.4 Outline of results

Just as in the well-known situation of the clamped plate, the scaled displacement $\mathbf{u}(\varepsilon)$ tends in Ω to a Kirchhoff-Love displacement. By extension we introduce

Definition 1.1 A displacement $\mathbf{u}$ in Ω^ε is called a *Kirchhoff-Love displacement* if there exists a displacement $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ in the mean surface ω (i.e. function of the in-plane variable x_* only) such that

$$\mathbf{u} = (\zeta_1 - \tilde{x}_3 \partial_1 \zeta_3, \zeta_2 - \tilde{x}_3 \partial_2 \zeta_3, \zeta_3).$$

$\boldsymbol{\zeta}$ is called the *generator* of $\mathbf{u}$, and the scaled displacement associated with $\mathbf{u}$ in Ω has exactly the same form with $\tilde{x}_3$ replaced by x_3 . The bending and membrane parts of $\mathbf{u}$ are respectively

$$\mathbf{u}_{\text{KL},b} = (-\tilde{x}_3 \partial_1 \zeta_3, -\tilde{x}_3 \partial_2 \zeta_3, \zeta_3) \quad \text{and} \quad \mathbf{u}_{\text{KL},m} = (\zeta_1, \zeta_2, 0). \quad (1.17)$$

■

We prove in this paper (see Theorems 5.1, 6.1, 7.1, 8.1) that the displacement $\mathbf{u}^\varepsilon$ can be expanded in the following way in the sense of asymptotic expansions*

$$\begin{aligned} \mathbf{u}^\varepsilon \simeq & \frac{1}{\varepsilon} \mathbf{u}_{\text{KL},b}^0 + \mathbf{u}_{\text{KL},m}^0 + \mathbf{u}_{\text{KL},b}^1 + \varepsilon (\mathbf{u}_{\text{KL},m}^1 + \mathbf{u}_{\text{KL},b}^2 + \tilde{\mathbf{v}}^1 + \boldsymbol{\varphi}^1) + \dots \\ & \dots + \varepsilon^k (\mathbf{u}_{\text{KL},m}^k + \mathbf{u}_{\text{KL},b}^{k+1} + \tilde{\mathbf{v}}^k + \boldsymbol{\varphi}^k) + \dots \end{aligned} \quad (1.18)$$

where

*This means that if a norm is fixed — for example the H^1 or the L^2 norm of the displacements, or the energy — then there exists a shift d such that for any k the norm of the difference between $\mathbf{u}^\varepsilon$ and the truncated series of (1.18) at the order k is estimated by $c_k \varepsilon^{k-d}$. More precise considerations about the estimates will be addressed in Part II.

- $\mathbf{u}_{\text{KL},b}^k$ and $\mathbf{u}_{\text{KL},m}^k$ are the bending and membrane parts of a Kirchhoff-Love displacement with generator $\boldsymbol{\zeta}^k = (\zeta_1^k, \zeta_2^k, \zeta_3^k)$;
- $\tilde{\mathbf{v}}^k = \tilde{\mathbf{v}}^k(x_*, \frac{\tilde{x}_3}{\varepsilon})$, i.e. does not depend on ε in the scaled domain Ω ;
- $\boldsymbol{\varphi}^k = \boldsymbol{\varphi}^k(\frac{r}{\varepsilon}, s, \frac{\tilde{x}_3}{\varepsilon})$ is a boundary layer profile.

The generators $(\zeta_1^k, \zeta_2^k) = \boldsymbol{\zeta}_*^k$ and ζ_3^k of the above Kirchhoff displacements are solutions of membrane and bending equations respectively, with boundary conditions on $\partial\omega$ involving, in each of the four investigated cases:

	Membrane part		Bending part	
①	ζ_n	ζ_s	ζ_3	$\partial_n \zeta_3$
②	ζ_n	$T_s^m(\boldsymbol{\zeta}_*)$	ζ_3	$\partial_n \zeta_3$
③	$T_n^m(\boldsymbol{\zeta}_*)$	ζ_s	ζ_3	$M_n(\zeta_3)$
④	$T_n^m(\boldsymbol{\zeta}_*)$	$T_s^m(\boldsymbol{\zeta}_*)$	ζ_3	$M_n(\zeta_3)$

Table 2. Boundary conditions for the Kirchhoff-Love displacements.

Here ζ_n and ζ_s are the normal and tangential components of $\boldsymbol{\zeta}_*$, T_n^m and T_s^m are the normal and tangential components of the traction (3.1) associated with the membrane operator, and M_n is the second order boundary operator (3.2a) associated with the bending operator. The mechanical interpretation of M_n states that it corresponds to the ‘Kirchhoff bending moment’ on the lateral side of the plate.

Unlike the case of the four other lateral boundary conditions, the boundary data for $\boldsymbol{\zeta}^0$ are all zero. All the boundary data for $\boldsymbol{\zeta}^1$ are special traces of $\boldsymbol{\zeta}^0$, according to the next table (we recall that κ is the curvature of $\partial\omega$)

	Membrane part		Bending part	
①	$\text{div}_* \boldsymbol{\zeta}_*^0$	0	0	$\Delta_* \zeta_3^0$
②	$\text{div}_* \boldsymbol{\zeta}_*^0$	$\partial_s \text{div}_* \boldsymbol{\zeta}_*^0$	0	$\Delta_* \zeta_3^0$
③	$\kappa^2 \zeta_n^0$	0	0	$\kappa^2 \partial_n \zeta_3^0$
④	$\kappa \text{div}_* \boldsymbol{\zeta}_*^0$	$\partial_s \text{div}_* \boldsymbol{\zeta}_*^0$	0	$(\kappa^2 + \varrho \partial_{ss}) \partial_n \zeta_3^0$

Table 3. Boundary data for $\boldsymbol{\zeta}^1$.

Here, ϱ depends only on λ and μ and we omit other multiplicative constants also depending only on the Lamé coefficients in front of each the boundary values: these multiplicative constants come from typical boundary layer profiles.

The first non-Kirchhoff displacement $\tilde{\mathbf{v}}^1$ is also completely determined by $\boldsymbol{\zeta}^0$, but in a way which does not depend on the lateral boundary conditions:

$$\tilde{\mathbf{v}}^1(x_*, x_3) = \frac{\lambda}{6(\lambda + 2\mu)} (0, 0, -6x_3 \text{div}_* \boldsymbol{\zeta}_*^0 + (3x_3^2 - 1) \Delta_* \zeta_3^0). \quad (1.19)$$

The first boundary layer profile φ^1 can be described as a sum of three terms in tensor product form in the variables s and (t, x_3) with $t = \frac{r}{\varepsilon}$:

$$\varphi^1 = \ell^m(s) \bar{\varphi}^m(t, x_3) + \ell^b(s) \bar{\varphi}^b(t, x_3) + \ell^s(s) \bar{\varphi}^s(t, x_3). \quad (1.20)$$

Here $\bar{\varphi}^m$, $\bar{\varphi}^b$ and $\bar{\varphi}^s$ are typical profiles whose components have special parities with respect to x_3 :

Components	$\bar{\varphi}^m$	$\bar{\varphi}^b$	$\bar{\varphi}^s$
Normal	even	odd	0
Horizontal tangential	0	0	odd
Vertical	odd	even	0

Table 4. Typical boundary layer profiles.

The functions ℓ are given as traces of ζ^0 along the boundary $\partial\omega$ according to the following table :

Case	ℓ^m	ℓ^b	ℓ^s
① and ②	$\text{div}_* \zeta_*^0$	$\Delta_* \zeta_3^0$	0
③	$\kappa \zeta_n^0$	$\kappa \partial_n \zeta_3^0$	0
④	$\text{div}_* \zeta_*^0$	$\kappa \partial_n \zeta_3^0$	$\partial_s(\partial_n \zeta_3^0)$

Table 5. Lateral traces coming up in the first boundary layer profile.

Note the presence of κ in front of the traces for the hard simple support case: the existence of boundary layer terms is linked to non zero curvature, compare also with [2]. The regularity of the main profiles is discussed in Part II.

2 INNER – OUTER EXPANSION ANSATZ

2.1 The Ansatz

Starting from the hard clamped plate situation [5, 6, 7], it is natural to propose an Ansatz for $\mathbf{u}(\varepsilon)$ combining a power series like $\sum_k \varepsilon^k \underline{\mathbf{u}}^k(x)$ (*outer expansion*) and a boundary layer series like $\sum_k \varepsilon^k \mathbf{w}^k(r\varepsilon^{-1}, s, x_3)$ (*inner expansion*), with r being the distance to $\partial\omega$ and s an arc-length coordinate in $\partial\omega$. In the sequel, t denotes $r\varepsilon^{-1}$.

We use also the splitting of coordinates, components and gradients in plane and vertical variables $x = (x_*, x_3)$, $\zeta = (\zeta_*, \zeta_3)$ and $\nabla = (\nabla_*, \partial_3)^\top$.

The Ansatz contains three types of terms:

- $\mathbf{u}_{\text{KL}}^k$: Kirchhoff-Love displacements with ‘generating functions’ $\zeta^k = (\zeta_*^k, \zeta_3^k)$, i.e. $\mathbf{u}_{\text{KL}}^k(x) = \left(\zeta_*^k(x_*) - x_3 \nabla_* \zeta_3^k(x_*), \zeta_3^k(x_*) \right)$,
- $\mathbf{v}^k$: displacements with zero mean value: $\forall x_* \in \overline{\omega}, \int_{-1}^{+1} \mathbf{v}^k(x_*, x_3) dx_3 = 0$,
- $\mathbf{w}^k$: exponentially decreasing profiles as $t \rightarrow +\infty$ and χ is a cut-off function equal to 1 in a neighborhood of $\partial\omega$,

and can be written as:

$$\mathbf{u}(\varepsilon)(x) \simeq \mathbf{u}_{\text{KL}}^0 + \varepsilon \mathbf{u}^1(x, \frac{r}{\varepsilon}) + \cdots + \varepsilon^k \mathbf{u}^k(x, \frac{r}{\varepsilon}) + \cdots \quad (2.1)$$

where

$$\begin{aligned} \mathbf{u}^1(x, t) &= \mathbf{u}_{\text{KL}}^1 + \chi(r) \mathbf{w}^1(t, s, x_3) & \text{with } w_3^1 &= 0, \\ \mathbf{u}^k(x, t) &= \mathbf{u}_{\text{KL}}^k + \mathbf{v}^k + \chi(r) \mathbf{w}^k(t, s, x_3) & \text{for } k &\geq 2. \end{aligned} \quad (2.2)$$

The links with expansion (1.18) on the thin plates are simply provided by the following relations

$$\mathbf{u}_{\text{KL},b}^k + \mathbf{u}_{\text{KL},m}^k = \mathbf{u}_{\text{KL}}^k, \quad \tilde{\mathbf{v}}^k = (\mathbf{v}_*^k, v_3^{k+1}), \quad \boldsymbol{\varphi}^k = (\mathbf{w}_*^k, w_3^{k+1}). \quad (2.3)$$

The determination of the asymptotics (2.1) can be split into two steps. The first one consists in finding *all* suitable power series

$$\underline{\mathbf{u}}(\varepsilon)(x) \simeq \underline{\mathbf{u}}^0(x) + \varepsilon \underline{\mathbf{u}}^1(x) + \cdots + \varepsilon^k \underline{\mathbf{u}}^k(x) + \cdots \quad (2.4)$$

which solve in the sense of asymptotic expansions the interior equations (1.13) in Ω and conditions (1.14) of traction on the horizontal sides $\Gamma_{\pm}$. We refer to MAZ’YA, NAZAROV & PLAMENEVSKII [13, Ch.15] for general developments relating to the structure of expansion (2.4).

We will see in the sequel that all the terms in the suitable series (2.4) are strictly determined *except* the only elliptic traces of the Kirchhoff-Love generators ζ^k . The second step which we will initiate in section 3, consists in finding the profiles $\mathbf{w}^k$ so that $\sum_k \varepsilon^k \mathbf{w}^k(r\varepsilon^{-1}, s, x_3)$ solves equations (1.13) in Ω with zero volume force, conditions (1.14) of zero traction and so that the lateral boundary conditions (1.15) are satisfied by the complete Ansatz. The outcome will be that the existence of *exponentially decaying* profiles are subordinated to the determination of the remaining degrees of freedom in the series (2.4). This will be made explicit in each of the four cases ① to ④. Here we leave the question of the estimates of $\mathbf{u}(\varepsilon)$ minus any truncated series from (2.1) because the results and the way of proving them are strictly similar to what is presented in [5, 6, 7]. A more detailed remark on the estimates will be given in Part II.

2.2 The algorithms of the outer expansion part

This section is devoted to the construction of the most general power series (2.4) solving (1.13)-(1.14). Let us introduce the two operators A and B which associate

to a displacement $\mathbf{u}$ in Ω a volume force in Ω and tractions on the horizontal sides on $\Gamma_{\pm}$ according to:

$$\begin{aligned} A\mathbf{u} &= \left(2\mu \partial_3 e_{\alpha 3}(\mathbf{u}) + \lambda \partial_{\alpha 3} u_3, (\lambda + 2\mu) \partial_{33} u_3; 2\mu e_{\alpha 3}(\mathbf{u})|_{\Gamma_{\pm}}, (\lambda + 2\mu) \partial_3 u_3|_{\Gamma_{\pm}} \right) \\ B\mathbf{u} &= \left((\lambda + \mu) \partial_{\alpha} \operatorname{div}_* \mathbf{u}_* + \mu \Delta_* u_{\alpha}, \lambda \partial_3 \operatorname{div}_* \mathbf{u}_* + 2\mu \partial_{\gamma} e_{\gamma 3}(\mathbf{u}); 0|_{\Gamma_{\pm}}, \lambda \operatorname{div}_* \mathbf{u}_*|_{\Gamma_{\pm}} \right) \end{aligned}$$

the first group of arguments being the in-plane volume forces, the second, the transverse volume force, and similarly for the tractions. Solving (1.13)-(1.14) by a power series (2.4) is equivalent to solve the system of equations

$$\begin{cases} A\mathbf{u}^k &= 0 & \text{for } k = 0, 1, \\ A\mathbf{u}^2 + B\mathbf{u}^0 &= \left(-f_{\alpha}, 0; g_{\alpha}^{\pm}|_{\Gamma_{\pm}}, 0|_{\Gamma_{\pm}} \right), \\ A\mathbf{u}^4 + B\mathbf{u}^2 &= \left(0, -f_3; 0|_{\Gamma_{\pm}}, g_3^{\pm}|_{\Gamma_{\pm}} \right), \\ A\mathbf{u}^k + B\mathbf{u}^{k-2} &= 0, & \text{for } k = 3 \text{ and } k \geq 5. \end{cases} \quad (2.5)$$

It is well known that the solutions of the problem $A\mathbf{u} = 0$ are the Kirchhoff-Love displacements. Thus $\mathbf{u}^0 = \mathbf{u}_{\text{KL}}^0$ and $\mathbf{u}^1 = \mathbf{u}_{\text{KL}}^1$, with generators ζ^0 and ζ^1 respectively, i.e.:

$$\mathbf{u}_{\text{KL}}^k(x) = \left(\zeta_*^k(x_*) - x_3 \nabla_* \zeta_3^k(x_*), \zeta_3^k(x_*) \right) \quad \text{for } k = 0, 1.$$

In order to solve the series of equations of *odd order* $A\mathbf{u}^k + B\mathbf{u}^{k-2} = 0$, let us introduce the operator V .

Definition 2.1 The operator $V : \zeta \mapsto V\zeta$ is defined from $\mathcal{C}^{\infty}(\overline{\omega})^3$ into $\mathcal{C}^{\infty}(\overline{\Omega})^3$ by

$$\begin{aligned} (V\zeta)_{\alpha} &= \bar{p}_2 \partial_{\alpha} \operatorname{div}_* \zeta_* + \bar{p}_3 \partial_{\alpha} \Delta_* \zeta_3 \\ (V\zeta)_3 &= \bar{p}_1 \operatorname{div}_* \zeta_* + \bar{p}_2 \Delta_* \zeta_3 \end{aligned} \quad (2.6)$$

with $\bar{p}_j$ for $j = 1, 2, 3$ the polynomials in the variable x_3 of degrees j defined as

$$\begin{aligned} \bar{p}_1(x_3) &= -\frac{\tilde{\lambda}}{2\mu} x_3, & \bar{p}_2(x_3) &= \frac{\tilde{\lambda}}{4\mu} \left(x_3^2 - \frac{1}{3} \right), \\ \bar{p}_3(x_3) &= \frac{1}{12\mu} \left((\tilde{\lambda} + 4\mu) x_3^3 - (5\tilde{\lambda} + 12\mu) x_3 \right). \end{aligned} \quad (2.7)$$

Here $\tilde{\lambda}$ denotes the ‘homogenized’ Lamé coefficient $2\lambda\mu(\lambda + 2\mu)^{-1}$. ■

Recall that Δ_* denotes the horizontal Laplacian $\partial_{11} + \partial_{22}$ and let Δ_* denote the diagonal horizontal Laplacian. Let L^m be the membrane operator (plane stress model)

$$L^m \zeta_* = \mu \Delta_* \zeta_* + (\tilde{\lambda} + \mu) \nabla_* \operatorname{div}_* \zeta_* \quad . \quad (2.8)$$

Then direct computations yield

Lemma 2.2 *Let ζ belong to $\mathcal{C}^\infty(\overline{\omega})^3$ and let $\mathbf{u}_{\text{KL}}$ be the associated Kirchhoff-Love displacement. Then the field $V\zeta$ is the unique solution with zero mean values on each fiber $x_* \times (-1, 1)$ of the problem*

$$A(V\zeta) + B(\mathbf{u}_{\text{KL}}) = \left(L^{\text{m}}\zeta_*, 0; 0 \Big|_{\Gamma_\pm}, 0 \Big|_{\Gamma_\pm} \right). \quad (2.9)$$

Then, if $L^{\text{m}}\zeta_*^1 = 0$, we can take $\underline{\mathbf{u}}^3 = \mathbf{u}_{\text{KL}}^3 + V\zeta^1$. In order to proceed, we remark that each component of $B(V\zeta)$ can be split into two parts, both of them being the product of a polynomial in x_3 and of $\Delta_* \text{div}_* \zeta_*$ or $\Delta_*^2 \zeta_3$, or a derivative of these expressions. Introducing the bending operator

$$L^{\text{b}}\zeta_3 = (\tilde{\lambda} + 2\mu)\Delta_*^2 \zeta_3, \quad (2.10)$$

we easily obtain that if $L^{\text{m}}\zeta_*$ and $L^{\text{b}}\zeta_3$ are zero, then $B(V\zeta)$ is zero, too. Thus, the odd part of the outer Ansatz is solved, since we obtain by an induction argument:

Proposition 2.3 *For any $k = 1, 3, 5, \dots$ let ζ^k be such that*

$$L^{\text{m}}\zeta_*^k = 0 \quad \text{and} \quad L^{\text{b}}\zeta_3^k = 0.$$

Then, setting for $k = 3, 5, \dots$

$$\underline{\mathbf{u}}^k = \mathbf{u}_{\text{KL}}^k + V\zeta^{k-2},$$

we obtain all the solutions of the odd order equations in system (2.5).

Let us consider now the equations of even order. The operator A is block triangular and its diagonal is made of ordinary Neumann problems on the interval $(-1, 1)$. So actually, in order to have solvability for these problems, compatibility conditions are required on the right-hand sides. Conversely, if the problems are solvable, the solutions are uniquely determined if we require that they have a mean value zero on each fiber $x_* \times (-1, 1)$ with $x_* \in \overline{\omega}$.

With $\underline{\mathbf{u}}^0 = \mathbf{u}_{\text{KL}}^0$, we will find $\underline{\mathbf{u}}^2$ being of the form $\mathbf{u}_{\text{KL}}^2 + V\zeta^0 + G(\mathbf{f}, \mathbf{g}^\pm)$, where G is another solution operator. But prior to this, we need two sorts of primitive of an integrable function u on the interval $(-1, +1)$:

Notation 2.4 *Let us introduce:*

- *The primitive of u with zero mean value on $(-1, +1)$*

$$\oint^{x_3} u \, dy_3 := \int_{-1}^{x_3} u(y_3) \, dy_3 - \frac{1}{2} \int_{-1}^{+1} \int_{-1}^{z_3} u(y_3) \, dy_3 \, dz_3,$$

- *The primitive of u which vanishes in -1 and 1 if u has a zero mean value on $(-1, +1)$ and which is even, resp. odd, if u is odd, resp. even*

$$\int^{y_3} u \, dz_3 := \frac{1}{2} \left(\int_{-1}^{y_3} u(z_3) \, dz_3 - \int_{y_3}^{+1} u(z_3) \, dz_3 \right).$$

Definition 2.5 The operator $G : (\mathbf{f}, \mathbf{g}^\pm) \mapsto G(\mathbf{f}, \mathbf{g}^\pm)$ is defined from $\mathcal{C}^\infty(\overline{\Omega})^3 \times \mathcal{C}^\infty(\overline{\omega})^6$ into $\mathcal{C}^\infty(\overline{\Omega})^3$ by

$$(G(\mathbf{f}, \mathbf{g}^\pm))_3 = 0$$

and

$$(G(\mathbf{f}, \mathbf{g}^\pm))_\alpha = \frac{1}{2\mu} \oint^{x_3} \left[-2 \int^{y_3} f_\alpha + (g_\alpha^+ - g_\alpha^- + \int_{-1}^{+1} f_\alpha) y_3 + g_\alpha^+ + g_\alpha^- \right] dy_3 .$$

■

The reason for the introduction of G is

Lemma 2.6 For any $(\mathbf{f}, \mathbf{g}^\pm) \in \mathcal{C}^\infty(\overline{\Omega})^3 \times \mathcal{C}^\infty(\overline{\omega})^6$, $G(\mathbf{f}, \mathbf{g}^\pm)$ is the unique solution with zero mean values on each fiber $x_* \times (-1, 1)$ of the problem

$$A(G(\mathbf{f}, \mathbf{g}^\pm)) = \left(-f_\alpha + \frac{1}{2} \left[\int_{-1}^{+1} f_\alpha dx_3 + g_\alpha^+ - g_\alpha^- \right], 0; g_\alpha^\pm|_{\Gamma_\pm}, 0|_{\Gamma_\pm} \right).$$

Now, we can see that if we set

$$\mathbf{R}_m^0(x_*) = -\frac{1}{2} \left[\int_{-1}^{+1} \mathbf{f}_*(x_*, x_3) dx_3 + \mathbf{g}_*^+(x_*) - \mathbf{g}_*^-(x_*) \right], \quad (2.11)$$

for any ζ_*^0 satisfying the membrane equation

$$L^m(\zeta_*^0) = \mathbf{R}_m^0, \quad (2.12)$$

the displacement $\underline{\mathbf{u}}^2 = \mathbf{u}_{\text{KL}}^2 + V\zeta^0 + G(\mathbf{f}, \mathbf{g}^\pm)$ solves the equation of order $k = 2$ of system (2.5). We denote by $\mathbf{v}^2 = V\zeta^0 + G$ its part with zero mean values on each fiber $x_* \times (-1, 1)$.

In order to go further in solving the even part of the Ansatz, we are going to introduce a residual operator $F = (F_*, F_3)$ and a new solution operator W .

Definition 2.7 (i) The operator $F : \mathbf{v} \mapsto F\mathbf{v} = (F_*\mathbf{v}, F_3\mathbf{v})$ is defined from $\mathcal{C}^\infty(\overline{\Omega})^3$ into $\mathcal{C}^\infty(\overline{\omega})^3$ by

$$(F_*\mathbf{v})_\alpha = \frac{\tilde{\lambda}}{2} \int_{-1}^{+1} \int^{y_3} \partial_{\alpha\gamma} e_{\gamma 3}(\mathbf{v}) dz_3 dy_3$$

and

$$F_3\mathbf{v} = \mu \int_{-1}^{+1} \partial_\gamma e_{\gamma 3}(\mathbf{v}) dy_3.$$

(ii) The operator $W : \mathbf{v} \mapsto W\mathbf{v}$ is defined from $\mathcal{C}^\infty(\overline{\Omega})^3$ into itself by

$$(W\mathbf{v})_3 = - \oint^{x_3} \left(\frac{\tilde{\lambda}}{2\mu} \operatorname{div}_* \mathbf{v}_* + \frac{\tilde{\lambda}}{\lambda} \int^{y_3} \partial_\gamma e_{\gamma 3}(\mathbf{v}) \right) dy_3$$

and

$$(W\mathbf{v})_\alpha = - \oint^{x_3} \left(\partial_\alpha (W\mathbf{v})_3 + \int^{y_3} \left(\frac{\lambda}{\mu} \partial_{\alpha 3} (W\mathbf{v})_3 + \frac{\lambda + \mu}{\mu} \partial_\alpha \operatorname{div}_* \mathbf{v}_* + \Delta_* v_\alpha \right) \right) dy_3 .$$

■

With these operators, we can prove

Lemma 2.8 *Let $\mathbf{v}$ in $\mathcal{C}^\infty(\overline{\Omega})^3$ be a displacement field with zero mean values on each fiber $x_* \times (-1, 1)$, $x_* \in \overline{\omega}$. Then $W\mathbf{v}$ has also zero mean values on each fiber $x_* \times (-1, 1)$ and solves the problem*

$$A(W\mathbf{v}) + B(\mathbf{v}) = \left(0, 0; \pm F_*(\mathbf{v})\Big|_{\Gamma_\pm}, \mp F_3(\mathbf{v})\Big|_{\Gamma_\pm}\right).$$

Now, it is natural to search for $\underline{\mathbf{u}}^4$ with the form $\mathbf{u}_{\text{KL}}^4 + V\boldsymbol{\zeta}^2 + W(V\boldsymbol{\zeta}^0 + G) + H$. In view of Lemmas 2.2 and 2.8, with such an Ansatz, H has to solve the problem

$$AH = \left(-L^m(\boldsymbol{\zeta}_*^2), -f_3; \mp F_*(V\boldsymbol{\zeta}^0 + G)\Big|_{\Gamma_\pm}, g_3^\pm \pm F_3(V\boldsymbol{\zeta}^0 + G)\Big|_{\Gamma_\pm}\right). \quad (2.13)$$

Thus, it is important to have more information about $F(V\boldsymbol{\zeta}^0 + G)$. It is not difficult to check:

Lemma 2.9 *For all $\boldsymbol{\zeta}$ in $\mathcal{C}^\infty(\overline{\omega})^3$ we have*

$$F_*(V\boldsymbol{\zeta}) = 0 \quad \text{and} \quad F_3(V\boldsymbol{\zeta}) = -\frac{1}{3}L^b\zeta_3.$$

Moreover, we have

$$F_3(G) = \frac{1}{2} \operatorname{div}_* \left[\int_{-1}^{+1} x_3 \mathbf{f}_* dx_3 + \mathbf{g}_*^+ + \mathbf{g}_*^- \right]. \quad (2.14)$$

Then there holds

Lemma 2.10 *Let R_b^0 be defined as*

$$R_b^0 = \frac{3}{2} \left[\int_{-1}^{+1} f_3 dx_3 + g_3^+ - g_3^- + \operatorname{div}_* \left(\int_{-1}^{+1} x_3 \mathbf{f}_* dx_3 + \mathbf{g}_*^+ + \mathbf{g}_*^- \right) \right], \quad (2.15)$$

and $\mathbf{R}_m^2$ be defined as

$$\mathbf{R}_m^2 = F_*(G) - \frac{\tilde{\lambda}}{4\mu} \nabla_* \left[\int_{-1}^{+1} x_3 f_3 dx_3 + g_3^+ + g_3^- \right]. \quad (2.16)$$

If the following conditions are satisfied

$$L^b(\zeta_3^0) = R_b^0 \quad \text{and} \quad L^m(\boldsymbol{\zeta}_*^2) = \mathbf{R}_m^2, \quad (2.17)$$

then equation (2.13) admits a unique solution $H = H(\mathbf{f}, \mathbf{g}^\pm)$ with zero mean values on each fiber $x_* \times (-1, 1)$ which is given by

$$H_3 = \frac{1}{2(\lambda + 2\mu)} \oint^{x_3} \left[(-2 \oint^{y_3} f_3) + g_3^+ + g_3^- \right] dy_3$$

and

$$H_\alpha = - \oint^{x_3} \left[\partial_\alpha H_3 + \frac{1}{\mu} y_3 F_*(G) + \frac{\lambda}{\mu} \oint^{y_3} \left\{ \partial_{\alpha 3} H_3 - \frac{1}{2} \int_{-1}^{+1} \partial_{\alpha 3} H_3 dz_3 \right\} \right] dy_3.$$

Thus, we have found $\underline{u}^4$ as $\underline{u}_{\text{KL}}^4 + \mathbf{v}^4$ where $\mathbf{v}^4$ has zero mean values on each fiber $x_* \times (-1, 1)$: $\mathbf{v}^4$ is given by $V\zeta^2 + W(V\zeta^0 + G) + H = V\zeta^2 + W\mathbf{v}^2 + H$.

Next, we search for a $\underline{u}^6$ with the form $\underline{u}_{\text{KL}}^6 + V\zeta^4 + W\mathbf{v}^4 + X$. In view of Lemmas 2.2 and 2.8, with such an Ansatz, X has to solve the problem

$$AX = \left(-L^m(\zeta_*^4), 0; \mp F_*(\mathbf{v}^4) \Big|_{\Gamma_{\pm}}, \pm F_3(\mathbf{v}^4) \Big|_{\Gamma_{\pm}} \right). \quad (2.18)$$

This problem is solvable (i) if $F_3(\mathbf{v}^4)$ is zero, which holds true if

$$L^b\zeta_3^2 = 3F_3(W\mathbf{v}^2 + H),$$

and (ii) if $L^m(\zeta_*^4) = F_*(\mathbf{v}^4)$, compare Lemma 2.10. Then $X = X(\zeta_*^4)$ solves equation (2.18), with the solution operator X defined as

Definition 2.11 For $\zeta_* \in \mathcal{C}^\infty(\overline{\omega})^2$, $X = X(\zeta_*)$ is defined as

$$X_3 = 0 \quad \text{and} \quad X_* = -2\tilde{\lambda}^{-1}\bar{p}_2 L^m(\zeta_*).$$

■

And from now on, the solution of the series of equations (2.5) is self-similar. Summarizing, we obtain by induction that every expansion (2.4) solving (1.13)-(1.14) can be described according to Table 6 below, where G and H are a condensed notation for $G(\mathbf{f}, \mathbf{g}^\pm)$ and $H(\mathbf{f}, \mathbf{g}^\pm)$ respectively and $\mathbf{R}_m^k$ and R_b^k are the prescribed values for $L^m(\zeta_*^k)$ and $L^b(\zeta_3^k)$ respectively (note that $\mathbf{R}_m^0$, $\mathbf{R}_m^2$ and R_b^0 are defined in (2.11), (2.16) and (2.15)). Here, the even order terms and the odd order ones are independent from each other. We will see later on that they are connected by the lateral boundary conditions via the boundary layer terms.

k	$\underline{u}^k$	$\mathbf{v}^k$	$\mathbf{x}^{k-2}$	$\mathbf{R}_m^k$	R_b^k
0	$\underline{u}_{\text{KL}}^0$	—	—	$\mathbf{R}_m^0$	R_b^0
2	$\underline{u}_{\text{KL}}^2 + \mathbf{v}^2$	$V\zeta^0 + \mathbf{x}^0$	G	$\mathbf{R}_m^2$	$3F_3(W\mathbf{v}^2 + H)$
4	$\underline{u}_{\text{KL}}^4 + \mathbf{v}^4$	$V\zeta^2 + \mathbf{x}^2$	$W\mathbf{v}^2 + H$	$F_*\mathbf{v}^4$	$3F_3(W\mathbf{v}^4 + X\zeta_*^4)$
$2\ell+2$	$\underline{u}_{\text{KL}}^{2\ell+2} + \mathbf{v}^{2\ell+2}$	$V\zeta^{2\ell} + \mathbf{x}^{2\ell}$	$W\mathbf{v}^{2\ell} + X\zeta_*^{2\ell}$	$F_*\mathbf{v}^{2\ell+2}$	$3F_3(W\mathbf{v}^{2\ell+2} + X\zeta_*^{2\ell+2})$
1	$\underline{u}_{\text{KL}}^1$	—	—	0	0
$2\ell+1$	$\underline{u}_{\text{KL}}^{2\ell+1} + \mathbf{v}^{2\ell+1}$	$V\zeta^{2\ell-1}$	—	0	0

Table 6. Algorithm formulas.

We emphasize that each term $\underline{u}^k$ in the algorithm is the sum of two terms $\underline{u}^k = \underline{u}_{\text{KL}}^k + \mathbf{v}^k$ with $\underline{u}_{\text{KL}}^k$ representing the general solution of homogeneous Neumann problems for ordinary differential equations over each fiber $x_* \times (-1, 1)$ and $\mathbf{v}^k$ being particular solutions of inhomogeneous ordinary Neumann problems across the thickness with mean value zero.

2.3 Formulas for the determined part of the displacements

The formulas in Table 6 giving the $\mathbf{v}^k$ yield in a straightforward way that

$$\begin{aligned} \mathbf{v}^{2k+1} &= V\zeta^{2k-1}, \\ \mathbf{v}^{2k+2} &= \sum_{\ell=0}^k W^\ell \circ V\zeta^{2(k-\ell)} + \sum_{\ell=0}^{k-2} W^\ell \circ X\zeta_*^{2(k-\ell)} \\ &\quad + W^k \circ G(\mathbf{f}, \mathbf{g}^\pm) + W^{k-1} \circ H(\mathbf{f}, \mathbf{g}^\pm) \end{aligned} \quad (2.19)$$

with the convention that $W^{-1} = 0$ and $W^0 = \text{Id}$.

Using the definitions of V and W , we can prove

Lemma 2.12 *For $\ell = 0, 1, \dots$, we have the following formulas for the iterates $W^\ell \circ V$*

$$\begin{aligned} (W^\ell \circ V\zeta)_\alpha &= \bar{r}_{2\ell+2} \partial_\alpha \Delta_*^\ell \text{div}_* \zeta_* + \bar{r}_{2\ell+3} \partial_\alpha \Delta_*^{\ell+1} \zeta_3 \\ (W^\ell \circ V\zeta)_3 &= \bar{q}_{2\ell+1} \Delta_*^\ell \text{div}_* \zeta_* + \bar{q}_{2\ell+2} \Delta_*^{\ell+1} \zeta_3 \end{aligned} \quad (2.20)$$

with $\bar{q}_j, \bar{r}_j$ the polynomials in the variable x_3 of degrees j and of parities j defined recursively as

$$\bar{q}_1 = \bar{p}_1, \quad \bar{q}_2 = \bar{p}_2, \quad \bar{r}_2 = \bar{p}_2, \quad \bar{r}_3 = \bar{p}_3,$$

with $\bar{p}_j$ for $j = 1, 2, 3$ the polynomials defined in (2.7), and

$$\begin{aligned} \bar{q}_j(x_3) &= - \oint^{x_3} \left(\frac{\tilde{\lambda}}{2\mu} \bar{r}_{j-1} + \frac{\tilde{\lambda}}{2\lambda} \int^{y_3} (\bar{q}_{j-2} + \bar{r}'_{j-1}) \right) dy_3, \quad \text{for } j \geq 3 \\ \bar{r}_j(x_3) &= - \oint^{x_3} \left(\bar{q}_{j-1} + \int^{y_3} \left(\frac{\lambda}{\mu} \bar{q}'_{j-1} + \frac{\lambda+2\mu}{\mu} \bar{r}_{j-2} \right) \right) dy_3, \quad \text{for } j \geq 4. \end{aligned} \quad (2.21)$$

Similarly, using the definition of X we are able to show

Lemma 2.13 *For $\ell = 0, 1, \dots$, we have the following formulas for the iterates $W^\ell \circ X$*

$$\begin{aligned} (W^\ell \circ X\zeta_*)_\alpha &= \bar{s}_{2\ell+2} \partial_\alpha \Delta_*^\ell \text{div}_* \zeta_* + \bar{t}_{2\ell+2} \Delta_*^{\ell+1} \zeta_\alpha \\ (W^\ell \circ X\zeta_*)_3 &= \bar{s}_{2\ell+1} \Delta_*^\ell \text{div}_* \zeta_* \end{aligned} \quad (2.22)$$

with $\bar{s}_j$ and $\bar{t}_j$ the polynomials in the variable x_3 of degrees j and of parities j defined recursively as

$$\bar{s}_1 = 0, \quad \bar{s}_2 = -\frac{\lambda+2\mu}{\lambda} \bar{p}_2 \quad \text{and} \quad \bar{t}_2 = -\frac{3\lambda+2\mu}{\lambda} \bar{p}_2$$

with $\bar{p}_2$ given in (2.7), and for $\ell \geq 1$:

$$\begin{aligned} \bar{s}_{2\ell+1}(x_3) &= - \oint^{x_3} \left(\frac{\tilde{\lambda}}{2\mu} (\bar{s}_{2\ell} + \bar{t}_{2\ell}) + \frac{\tilde{\lambda}}{2\lambda} \int^{y_3} (\bar{s}_{2\ell-1} + \bar{s}'_{2\ell} + \bar{t}'_{2\ell}) \right) dy_3, \\ \bar{s}_{2\ell+2}(x_3) &= - \oint^{x_3} \left(\bar{s}_{2\ell+1} + \int^{y_3} \left(\frac{\lambda}{\mu} \bar{s}'_{2\ell+1} + \frac{\lambda+\mu}{\mu} (\bar{s}_{2\ell} + \bar{t}_{2\ell}) + \bar{s}_{2\ell} \right) \right) dy_3, \\ \bar{t}_{2\ell+2}(x_3) &= - \oint^{x_3} \left(\int^{y_3} \bar{t}_{2\ell} \right) dy_3. \end{aligned} \quad (2.23)$$

Condensing $G(\mathbf{f}, \mathbf{g}^\pm)$ into G and $H(\mathbf{f}, \mathbf{g}^\pm)$ into H , we obtain the following formulas for the first $\mathbf{v}^k$ (k even).

$$\begin{aligned} v_\alpha^2 &= \bar{p}_2 \partial_\alpha \operatorname{div}_* \zeta_*^0 + \bar{p}_3 \partial_\alpha \Delta_* \zeta_3^0 + G_\alpha \\ v_3^2 &= \bar{p}_1 \operatorname{div}_* \zeta_*^0 + \bar{p}_2 \Delta_* \zeta_3^0, \end{aligned} \quad (2.24)$$

$$\begin{aligned} v_\alpha^4 &= \bar{p}_2 \partial_\alpha \operatorname{div}_* \zeta_*^2 + \bar{p}_3 \partial_\alpha \Delta_* \zeta_3^2 + \bar{r}_4 \partial_\alpha \Delta_* \operatorname{div}_* \zeta_*^0 + \bar{r}_5 \partial_\alpha \Delta_*^2 \zeta_3^0 + (WG + H)_\alpha \\ v_3^4 &= \bar{p}_1 \operatorname{div}_* \zeta_*^2 + \bar{p}_2 \Delta_* \zeta_3^2 + \bar{q}_3 \Delta_* \operatorname{div}_* \zeta_*^0 + \bar{q}_4 \Delta_*^2 \zeta_3^0 + (WG + H)_3. \end{aligned} \quad (2.25)$$

3 THE PRINCIPLES OF CONSTRUCTION OF THE INNER EXPANSION PART

After the construction of the most general power series (2.4) solving (1.13)-(1.14), we see that the only remaining degrees of freedom are the elliptic traces of the Kirchhoff-Love generators ζ^k : that means two traces of ζ_*^k covering the membrane operator L^m (2.8) and two traces of ζ_3^k covering the bending operator L^b (2.10).

3.1 The membrane and bending boundary conditions

At this point, it is important to write down the Dirichlet and Neumann conditions associated to the membrane and bending operators.

(i) MEMBRANE. The bilinear form associated with the membrane operator L^m , cf (2.8) is:

$$a(\zeta_*, \eta_*) = \int_\omega \tilde{\lambda} e_{\alpha\alpha}(\zeta_*) e_{\beta\beta}(\eta_*) + 2\mu e_{\alpha\beta}(\zeta_*) e_{\alpha\beta}(\eta_*)$$

with the homogenized Lamé coefficient $\tilde{\lambda} = 2\lambda\mu(\lambda + 2\mu)^{-1}$. In normal and tangential components, cf (1.8)

$$\zeta_n = n_1 \zeta_1 + n_2 \zeta_2 \quad \text{and} \quad \zeta_s = (1 - \frac{r}{R})(n_2 \zeta_1 - n_1 \zeta_2),$$

the Dirichlet traces are simply (ζ_n, ζ_s) on $\partial\omega$, and the Neumann traces are

$$T_n^m(\zeta_*) = \tilde{\lambda} \operatorname{div}_* \zeta_* + 2\mu \partial_n \zeta_n, \quad (3.1a)$$

$$T_s^m(\zeta_*) = \mu(\partial_s \zeta_n + \partial_n \zeta_s + \frac{2}{R} \zeta_s), \quad (3.1b)$$

where ∂_n and ∂_s are defined in (1.7).

(ii) BENDING. The bilinear form associated with the bending operator L^b , cf (2.10) is:

$$b(\zeta_3, \eta_3) = \int_\omega \tilde{\lambda} \partial_{\alpha\alpha} \zeta_3 \partial_{\beta\beta} \eta_3 + 2\mu \partial_{\alpha\beta} \zeta_3 \partial_{\alpha\beta} \eta_3.$$

Its Dirichlet traces are ζ_3 and $\partial_n \zeta_3$ on $\partial\omega$, whereas the Neumann traces are

$$M_n(\zeta_3) = \tilde{\lambda} \Delta_* \zeta_3 + 2\mu \partial_{nn} \zeta_3, \quad (3.2a)$$

$$N_n(\zeta_3) = (\tilde{\lambda} + 2\mu) \partial_n \Delta_* \zeta_3 + 2\mu \partial_s (\partial_n + \frac{1}{R}) \partial_s \zeta_3. \quad (3.2b)$$

3.2 The equations of the inner expansion

As will be seen in detail for each case, the elliptic traces of the Kirchhoff-Love generators ζ^k are determined along with the computation of the boundary layer terms $\mathbf{w}^k$. To this aim, we notice that the boundary layer Ansatz, namely $\sum_{k \geq 1} \varepsilon^k \mathbf{w}^k$ must satisfy the equations (1.13) inside Ω with vanishing right hand side and the boundary conditions (1.14) of zero traction on the horizontal faces of Ω , and must compensate for the lateral boundary conditions of the power series $\sum_{k \geq 0} \varepsilon^k \underline{\mathbf{u}}^k$ so that the lateral boundary conditions (1.15) are fulfilled. We present in this section some common features of all problems.

(i) LATERAL BOUNDARY CONDITIONS. In order to obtain the relations which have to be satisfied by the inner part of the expansion, we evaluate the boundary conditions for a displacement $\mathbf{u}$ of the form

$$\mathbf{u}(\varepsilon)(x) = \underline{\mathbf{u}}(x) + (\varphi_*, \varepsilon \varphi_3) \left(\frac{r}{\varepsilon}, s, x_3 \right), \quad (3.3)$$

where $\underline{\mathbf{u}} = \sum_{k \geq 0} \varepsilon^k \underline{\mathbf{u}}^k$ and $\varphi = \sum_{k \geq 1} \varepsilon^k \varphi^k$.

The form of the boundary layer term $(\varphi_*, \varepsilon \varphi_3)$ is suggested by considerations about the symmetries of the elasticity operators: indeed we return with φ to the homogeneity of the original unknown $\mathbf{u}^\varepsilon$. Such a form allows a sort of conservation of each term of the strain tensor (possibly multiplied by a power of ε). We denote by φ_t the normal component of φ .

For $\mathbf{u}$ of the form (3.3), the formulas for the lateral Dirichlet conditions are obvious, and the lateral Neumann conditions can be written with the help of the boundary operators acting on the profiles φ

$$\begin{aligned} T_t^{(0)}(\varphi) &= \lambda \partial_3 \varphi_3 + (\lambda + 2\mu) \partial_t \varphi_t, & T_t^{(1)}(\varphi) &= \lambda (\partial_s \varphi_s - \frac{1}{R} \varphi_t), \\ T_s^{(0)}(\varphi) &= \mu \partial_t \varphi_s, & T_s^{(1)}(\varphi) &= \mu (\partial_s \varphi_t + \frac{2}{R} \varphi_s), \\ T_3^{(0)}(\varphi) &= \mu (\partial_t \varphi_3 + \partial_3 \varphi_t). \end{aligned} \quad (3.4)$$

Thus, we can write the components of the lateral traction, cf (1.16), as

$$T_n(\varepsilon) = \varepsilon T_t^{(0)}(\varphi) + \varepsilon^2 T_t^{(1)}(\varphi) + \lambda \partial_3 \underline{\mathbf{u}}_3 + \varepsilon^2 \left(\lambda \operatorname{div}_* \underline{\mathbf{u}}_* + 2\mu \partial_n \underline{\mathbf{u}}_n \right) \quad (3.5a)$$

$$T_s(\varepsilon) = \varepsilon T_s^{(0)}(\varphi) + \varepsilon^2 T_s^{(1)}(\varphi) + \varepsilon^2 \mu \left(\partial_s \underline{\mathbf{u}}_n + \partial_n \underline{\mathbf{u}}_s + \frac{2}{R} \underline{\mathbf{u}}_s \right) \quad (3.5b)$$

$$T_3(\varepsilon) = T_3^{(0)}(\varphi) + \mu (\partial_n \underline{\mathbf{u}}_3 + \partial_3 \underline{\mathbf{u}}_n). \quad (3.5c)$$

(ii) INTERIOR EQUATIONS. In variables (t, s, x_3) and unknowns

$$\varphi = (\varphi_t, \varphi_s, \varphi_3) \sim \left(\mathbf{w}_*, \frac{1}{\varepsilon} w_3 \right)$$

the interior equations (1.13) for $\mathbf{w}$ become

$$\mathcal{B}(\varepsilon; t, s; \partial_t, \partial_s, \partial_3) \varphi = 0.$$

In order to give the expression of $\mathcal{B}(\varepsilon)$, we introduce the notation

$$\rho = R(s) - r = R(s) - \varepsilon t$$

which is the curvature radius in s of the curve $\{x_* \in \omega, \text{dist}(x_*, \partial\omega) = r\}$. The three components $(\mathcal{B}\varphi)_t$, $(\mathcal{B}\varphi)_s$ and $(\mathcal{B}\varphi)_3$ of $\mathcal{B}\varphi$ are given by

$$\begin{aligned} (\mathcal{B}\varphi)_t &= \mu(\partial_{tt}\varphi_t + \partial_{33}\varphi_t) + (\lambda + \mu)\partial_t(\partial_t\varphi_t + \partial_3\varphi_3) \\ &\quad + \varepsilon\left(-\mu\left(\frac{1}{\rho}\partial_t\varphi_t\right) + (\lambda + \mu)\partial_t\left(-\frac{1}{\rho}\varphi_t + \frac{R}{\rho}\partial_s\left(\frac{R}{\rho}\varphi_s\right)\right)\right) \\ &\quad + \varepsilon^2\mu\left(-\frac{1}{\rho^2}\varphi_t + \left(\frac{R}{\rho}\partial_s\right)^2\varphi_t + \frac{R}{\rho}\partial_s\left(\frac{R}{\rho^2}\varphi_s\right) + \frac{R}{\rho^2}\partial_s\left(\frac{R}{\rho}\varphi_s\right)\right) \end{aligned} \quad (3.6a)$$

$$\begin{aligned} (\mathcal{B}\varphi)_s &= \mu\left(\partial_{tt}\left(\frac{R}{\rho}\varphi_s\right) + \partial_{33}\left(\frac{R}{\rho}\varphi_s\right)\right) \\ &\quad + \varepsilon\left(-\mu\left(\frac{1}{\rho}\partial_t\left(\frac{R}{\rho}\varphi_s\right)\right) + (\lambda + \mu)\frac{R}{\rho}\partial_s(\partial_t\varphi_t + \partial_3\varphi_3)\right) \\ &\quad + \varepsilon^2\left(\mu\left(-\frac{R}{\rho^3}\varphi_s + \left(\frac{R}{\rho}\partial_s\right)^2\left(\frac{R}{\rho}\varphi_s\right) - \frac{R}{\rho}\partial_s\left(\frac{1}{\rho}\varphi_t\right) - \frac{R}{\rho^2}\partial_s\varphi_t\right)\right. \\ &\quad \left.+ (\lambda + \mu)\frac{R}{\rho}\partial_s\left(-\frac{1}{\rho}\varphi_t + \frac{R}{\rho}\partial_s\left(\frac{R}{\rho}\varphi_s\right)\right)\right) \end{aligned} \quad (3.6b)$$

$$\begin{aligned} (\mathcal{B}\varphi)_3 &= \mu(\partial_{tt}\varphi_3 + \partial_{33}\varphi_3) + (\lambda + \mu)\partial_3(\partial_t\varphi_t + \partial_3\varphi_3) \\ &\quad + \varepsilon\left(-\mu\left(\frac{1}{\rho}\partial_t\varphi_3\right) + (\lambda + \mu)\partial_3\left(-\frac{1}{\rho}\varphi_t + \frac{R}{\rho}\partial_s\left(\frac{R}{\rho}\varphi_s\right)\right)\right) \\ &\quad + \varepsilon^2\mu\left(\left(\frac{R}{\rho}\partial_s\right)^2\varphi_3\right) \end{aligned} \quad (3.6c)$$

A Taylor expansion at $t = 0$ of $\rho^{-1} = (R - \varepsilon t)^{-1}$ yields an asymptotic expansion of $\mathcal{B}$ in a power series of ε :

$$\mathcal{B} \sim \mathcal{B}^{(0)} + \varepsilon\mathcal{B}^{(1)} + \dots \varepsilon^k\mathcal{B}^{(k)} + \dots \quad (3.7)$$

where the $\mathcal{B}^{(k)}(t, s; \partial_t, \partial_s, \partial_3)$ are partial differential systems of order 2 with polynomial coefficients in t independent from ε . As they are of constant use, we give here the expressions for $\mathcal{B}^{(0)}$ and $\mathcal{B}^{(1)}$:

$$\begin{aligned} (\mathcal{B}^{(0)}\varphi)_t &= \mu(\partial_{tt}\varphi_t + \partial_{33}\varphi_t) + (\lambda + \mu)\partial_t(\partial_t\varphi_t + \partial_3\varphi_3) \\ (\mathcal{B}^{(0)}\varphi)_s &= \mu(\partial_{tt}\varphi_s + \partial_{33}\varphi_s) \\ (\mathcal{B}^{(0)}\varphi)_3 &= \mu(\partial_{tt}\varphi_3 + \partial_{33}\varphi_3) + (\lambda + \mu)\partial_3(\partial_t\varphi_t + \partial_3\varphi_3) \end{aligned} \quad (3.8)$$

and, with the curvature $\kappa = \frac{1}{R}$:

$$\begin{aligned} (\mathcal{B}^{(1)}\varphi)_t &= -\mu\kappa\partial_t\varphi_t + (\lambda + \mu)\partial_t(-\kappa\varphi_t + \partial_s\varphi_s) \\ (\mathcal{B}^{(1)}\varphi)_s &= \mu\kappa(\partial_{tt}(t\varphi_s) + \partial_{33}(t\varphi_s)) - \mu\kappa\partial_t\varphi_s + (\lambda + \mu)\partial_s(\partial_t\varphi_t + \partial_3\varphi_3) \\ (\mathcal{B}^{(1)}\varphi)_3 &= -\mu\kappa\partial_t\varphi_3 + (\lambda + \mu)\partial_3(-\kappa\varphi_t + \partial_s\varphi_s). \end{aligned} \quad (3.9)$$

Thus, the interior equation $\mathcal{B}(\varepsilon)\varphi = 0$ can be written as

$$\mathcal{B}^{(0)}\varphi + \varepsilon\mathcal{B}^{(1)}\varphi + \dots\varepsilon^k\mathcal{B}^{(k)}\varphi + \dots \sim 0. \quad (3.10)$$

(iii) HORIZONTAL BOUNDARY CONDITIONS. The boundary conditions on the horizontal sides $x_3 = \pm 1$ are, cf (1.5)

$$\mu(\partial_3\varphi_t + \partial_t\varphi_3) = 0, \quad (3.11a)$$

$$\mu\partial_3\varphi_s + \varepsilon\mu\partial_s\varphi_3 = 0, \quad (3.11b)$$

$$(\lambda + 2\mu)\partial_3\varphi_3 + \lambda\partial_t\varphi_t + \varepsilon\lambda\left(-\frac{1}{\rho}\varphi_t + \frac{R}{\rho}\partial_s\left(\frac{R}{\rho}\varphi_s\right)\right) = 0. \quad (3.11c)$$

Similarly to the interior equations, we can develop the horizontal boundary conditions $\mathcal{G}$ (3.11) in powers of ε :

$$\mathcal{G} \sim \mathcal{G}^{(0)} + \varepsilon\mathcal{G}^{(1)} + \dots\varepsilon^k\mathcal{G}^{(k)} + \dots \quad (3.12)$$

where the $\mathcal{G}^{(k)}(t, s; \partial_t, \partial_s, \partial_3)$ are partial differential systems of order 1 with polynomial coefficients in t . The expressions for $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are:

$$\begin{aligned} (\mathcal{G}^{(0)}\varphi)_t &= \mu(\partial_3\varphi_t + \partial_t\varphi_3), & (\mathcal{G}^{(1)}\varphi)_t &= 0, \\ (\mathcal{G}^{(0)}\varphi)_s &= \mu\partial_3\varphi_s, & (\mathcal{G}^{(1)}\varphi)_s &= \mu\partial_s\varphi_3, \\ (\mathcal{G}^{(0)}\varphi)_3 &= (\lambda + 2\mu)\partial_3\varphi_3 + \lambda\partial_t\varphi_t, & (\mathcal{G}^{(1)}\varphi)_3 &= \lambda(-\kappa\varphi_t + \partial_s\varphi_s). \end{aligned} \quad (3.13)$$

Thus, the horizontal boundary conditions $\mathcal{G}(\varepsilon)\varphi = 0$ can be written as

$$\mathcal{G}^{(0)}\varphi + \varepsilon\mathcal{G}^{(1)}\varphi + \dots\varepsilon^k\mathcal{G}^{(k)}\varphi + \dots \sim 0. \quad (3.14)$$

3.3 The recursive equations

Assuming that $\sum_k \varepsilon^k \underline{\mathbf{u}}^k$ already fulfills the relations in Table 6, we determine now the equations satisfied by the profiles φ^k and the remaining conditions satisfied by the displacements $\underline{\mathbf{u}}^k$ so that

$$\sum_{k \geq 0} \varepsilon^k \underline{\mathbf{u}}^k + \sum_{k \geq 1} \varepsilon^k (\varphi_*^k, \varepsilon\varphi_3^k) \quad (3.15)$$

satisfies equations (1.13)-(1.15).

(i) INTERIOR EQUATIONS. (3.10) yields that

$$\forall k \geq 0, \quad \sum_{\ell=0}^k \mathcal{B}^{(\ell)} \boldsymbol{\varphi}^{k-\ell} = 0, \quad (3.16)$$

which guarantees (1.13) for the whole expansion (3.15).

(ii) HORIZONTAL BOUNDARY CONDITIONS. (3.14) yields that

$$\forall k \geq 0, \quad \sum_{\ell=0}^k \mathcal{G}^{(\ell)} \boldsymbol{\varphi}^{k-\ell} = 0, \quad (3.17)$$

which guarantees (1.14) for the whole expansion (3.15).

(iii) LATERAL DIRICHLET BOUNDARY CONDITIONS. Let $\sum_k \varepsilon^k D_n^k$, $\sum_k \varepsilon^k D_s^k$ and $\sum_k \varepsilon^k D_3^k$ be the normal, tangential and vertical components of the lateral Dirichlet traces of the series (3.15). The lateral Dirichlet boundary conditions then read

$$\forall k \geq 0, \quad D_n^k = 0 \text{ if } n \in A, \quad D_s^k = 0 \text{ if } s \in A, \quad D_3^k = 0 \text{ if } 3 \in A, \quad (3.18)$$

which immediately yields the Dirichlet conditions for the whole expansion (3.15).

For the terms D^k , we have

$$D_n^0 = \underline{u}_n^0, \quad D_s^0 = \underline{u}_s^0, \quad D_3^0 = \underline{u}_3^0, \quad D_3^1 = \underline{u}_3^1, \quad (3.19)$$

and for $k \geq 1$

$$D_n^k = \varphi_t^k + \underline{u}_n^k, \quad (3.20a)$$

$$D_s^k = \varphi_s^k + \underline{u}_s^k, \quad (3.20b)$$

$$D_3^{k+1} = \varphi_3^k + \underline{u}_3^{k+1}. \quad (3.20c)$$

(iv) LATERAL NEUMANN BOUNDARY CONDITIONS. Let $\sum_k \varepsilon^k T_n^k$, $\sum_k \varepsilon^k T_s^k$ and $\sum_k \varepsilon^k T_3^k$ be the normal, tangential and vertical components of the lateral Neumann traces of the series (3.15). The lateral Neumann boundary conditions then read

$$\forall k \geq 0, \quad T_n^k = 0 \text{ if } n \in B, \quad T_s^k = 0 \text{ if } s \in B, \quad T_3^k = 0 \text{ if } 3 \in B, \quad (3.21)$$

which immediately yields the Neumann conditions for the whole expansion (3.15).

Let us evaluate the terms T^k . To that aim, we rely on the following formulas for $\underline{u}^k$, cf Table 6, either $\underline{u}^k = \mathbf{u}_{\text{KL}}^k + \mathbf{v}^k$, i.e.

$$\underline{u}_n^k = \zeta_n^k - x_3 \partial_n \zeta_3^k + v_n^k, \quad (3.22a)$$

$$\underline{u}_s^k = \zeta_s^k - x_3 \partial_s \zeta_3^k + v_s^k, \quad (3.22b)$$

$$\underline{u}_3^k = \zeta_3^k + v_3^k, \quad (3.22c)$$

or $\underline{u}^k = \mathbf{u}_{\text{KL}}^k + V \zeta^{k-2} + \mathbf{x}^{k-2}$, i.e.

$$\underline{u}_n^k = \zeta_n^k - x_3 \partial_n \zeta_3^k + \bar{p}_2 \partial_n \operatorname{div}_* \zeta_*^{k-2} + \bar{p}_3 \partial_n \Delta_* \zeta_3^{k-2} + \mathbf{x}_n^{k-2}, \quad (3.23a)$$

$$\underline{u}_s^k = \zeta_s^k - x_3 \partial_s \zeta_3^k + \bar{p}_2 \partial_s \operatorname{div}_* \zeta_*^{k-2} + \bar{p}_3 \partial_s \Delta_* \zeta_3^{k-2} + \mathbf{x}_s^{k-2}, \quad (3.23b)$$

$$\underline{u}_3^k = \zeta_3^k + \bar{p}_1 \operatorname{div}_* \zeta_*^{k-2} + \bar{p}_2 \Delta_* \zeta_3^{k-2} + \mathbf{x}_3^{k-2}, \quad (3.23c)$$

where $\bar{p}_1, \bar{p}_2, \bar{p}_3$ are introduced in (2.7).

Thus, we find

$$T_n^0 = 0, \quad T_n^1 = 0, \quad T_s^0 = 0, \quad T_s^1 = 0, \quad T_3^0 = 0, \quad (3.24)$$

and for $k \geq 1$, cf (3.1), (3.2), (3.4):

$$T_n^{k+1} = T_t^{(0)}(\varphi^k) + T_t^{(1)}(\varphi^{k-1}) + T_n^m(\zeta_*^{k-1}) - x_3 M_n(\zeta_3^{k-1}) \\ + \lambda \partial_3 \mathbf{x}_3^{k-1} + \lambda \operatorname{div}_* \mathbf{v}_*^{k-1} + 2\mu \partial_n v_n^{k-1} \quad (3.25a)$$

$$T_s^{k+1} = T_s^{(0)}(\varphi^k) + T_s^{(1)}(\varphi^{k-1}) + T_s^m(\zeta_*^{k-1}) - 2\mu x_3 (\partial_n + \frac{1}{R}) \partial_s \zeta_3^{k-1} \\ + \mu \left(\partial_s v_n^{k-1} + \partial_n v_s^{k-1} + \frac{2}{R} v_s^{k-1} \right) \quad (3.25b)$$

$$T_3^k = T_3^{(0)}(\varphi^k) + \mu(\bar{p}_2 + \bar{p}_3') \partial_n \Delta_* \zeta_3^{k-2} \\ + \mu(\partial_n \mathbf{x}_3^{k-2} + \partial_3 \mathbf{x}_n^{k-2}). \quad (3.25c)$$

3.4 Solving the inner expansion

According to the calculations of the previous subsection, to solve the problem with the Ansatz (3.15), it remains to find a sequence of profiles $(\varphi^k)_k$ and a sequence of Kirchhoff-Love generators $(\zeta^k)_k$ such that (3.16), (3.17), (3.18) and (3.21) hold.

Let us consider now the profiles φ^k for $k \geq 1$ as main unknowns. In view of (3.16), (3.17) and (3.20) and (3.25), we see that the sequence of problems satisfied by the φ^k can be written in a recursive way: for each $k \geq 1$ the profile φ^k has to solve the equation

$$\mathcal{B}_{\textcircled{1}}(\varphi^k) = (\mathbf{f}^k; \mathbf{g}^k; \mathbf{h}^k), \quad (3.26)$$

where

- $\mathcal{B}_{\textcircled{1}}$ is the operator $\mathcal{B}^{(0)}$ inside the domain, the traction operator $\mathcal{G}^{(0)}$ on the horizontal sides, the Dirichlet traces on the lateral side for $a \in A_{\textcircled{1}}$ and the Neumann traces on the lateral side for $b \in B_{\textcircled{1}}$,
- $\mathbf{f}^k$ and $\mathbf{g}^k$ are the following functions of the previous profiles

$$\mathbf{f}^k = - \sum_{\ell=1}^k \mathcal{B}^{(\ell)} \varphi^{k-\ell} \quad \text{and} \quad \mathbf{g}^k = - \sum_{\ell=1}^k \mathcal{G}^{(\ell)} \varphi^{k-\ell}, \quad (3.27)$$

so that (3.16)-(3.17) is solved, and $\mathbf{h}^k$ involves previous profiles as well and certain traces of the Kirchhoff-Love generators ζ^ℓ according to (3.18)-(3.25).

An important point is now to note that neither $\mathcal{B}^{(0)}$, nor $\mathcal{G}^{(0)}$, nor the lateral trace operators of $\mathcal{B}_{\textcircled{1}}$ contain any derivative with respect to the tangential variable s . Thus, the equations (3.26) can be solved in the variables $t \in \mathbb{R}^+$ and $x_3 \in (-1, 1)$, the role of s being only that of a parameter. So we introduce the half-strip

$$\Sigma^+ = \left\{ (t, x_3); \quad 0 < t, \quad -1 < x_3 < 1 \right\}. \quad (3.28)$$

Its boundary has two horizontal parts $\gamma_{\pm} = \mathbb{R}^+ \times \{x_3 = \pm 1\}$ and a lateral part

$$\gamma_0 = \{(t, x_3); \quad t = 0, \quad -1 < x_3 < 1\}. \quad (3.29)$$

Thus, we have

$$\mathcal{B}_{\mathfrak{I}}(\varphi) = (\mathbf{f}; \mathbf{g}; \mathbf{h}) \iff \begin{cases} \mathcal{B}^{(0)}(\varphi) = \mathbf{f}, & \text{in } \Sigma^+, \\ \mathcal{G}^{(0)}(\varphi) = \mathbf{g}, & \text{on } \gamma_{\pm}, \\ \varphi_a = \mathbf{h}_a, & \text{on } \gamma_0, \quad \forall a \in A_{\mathfrak{I}}, \\ T_b^{(0)}(\varphi) = \mathbf{h}_b, & \text{on } \gamma_0, \quad \forall b \in B_{\mathfrak{I}}. \end{cases} \quad (3.30)$$

Essential is the possibility of finding *exponentially decreasing* solutions when $\mathbf{f}$ and $\mathbf{g}$ have the same property. This is what we start to investigate in the next section.

4 EXPONENTIALLY DECAYING PROFILES IN A HALF-STRIP

4.1 General principles

The properties of the operators $\mathcal{B}_{\mathfrak{I}}$ are closely linked to those of the corresponding operator $\mathcal{B}$ on the full strip $\Sigma := \mathbb{R} \times (-1, 1)$, defined as $\mathcal{B}(\varphi) = (\mathbf{f}; \mathbf{g})$ with $\mathbf{f} = \mathcal{B}^{(0)}(\varphi)$ in Σ and $\mathbf{g} = \mathcal{G}^{(0)}(\varphi)$ on $\mathbb{R} \times \{x_3 = \pm 1\}$, see also NAZAROV & PLAMENEVSKII [17, Ch.5].

Let $\mathcal{P}$ be the space of polynomial displacements $\mathbf{Z}$ satisfying $\mathcal{B}(\mathbf{Z}) = 0$. Computations like those of MIELKE in [15] yield that $\mathcal{P}$ has eight dimensions and that a basis of $\mathcal{P}$ is given by the following polynomial displacements $\mathbf{Z}^{[1]}, \dots, \mathbf{Z}^{[8]}$

$$\begin{aligned} \mathbf{Z}^{[1]} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \mathbf{Z}^{[2]} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \mathbf{Z}^{[3]} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \mathbf{Z}^{[4]} &= \begin{pmatrix} -x_3 \\ 0 \\ t \end{pmatrix} \\ \mathbf{Z}^{[5]} &= \begin{pmatrix} t \\ 0 \\ \bar{p}_1 \end{pmatrix} & \mathbf{Z}^{[6]} &= \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} & \mathbf{Z}^{[7]} &= \begin{pmatrix} -2tx_3 \\ 0 \\ t^2 + 2\bar{p}_2 \end{pmatrix} \\ \mathbf{Z}^{[8]} &= \begin{pmatrix} -3t^2x_3 + 6\bar{p}_3 \\ 0 \\ t^3 + 6t\bar{p}_2 \end{pmatrix} \end{aligned}$$

where $\bar{p}_1(x_3)$, $\bar{p}_2(x_3)$, $\bar{p}_3(x_3)$ are the polynomials previously introduced in (2.7).

Let us introduce weighted spaces H_{η}^m on the half-strip Σ^+ : for $\eta > 0$, their elements are exponentially decreasing as $t \rightarrow \infty$:

Definition 4.1 Let $\eta \in \mathbb{R}$. For $m \geq 0$ let $H_{\eta}^m(\Sigma^+)$ be the space of functions v such that $e^{\eta t}v$ belongs to $H^m(\Sigma^+)$. We also denote $H_{\eta}^0(\Sigma^+)$ by $L_{\eta}^2(\Sigma^+)$. Similar definitions hold for $\mathbb{R}^+$. ■

Like in [7, Lemmas 4.10 & 4.11], we have, with η_0 the smallest exponent arising from the Papkovitch-Fadle eigenfunctions, compare PAPKOVICH [19] for early reference and GREGORY & WAN [11]:

Lemma 4.2 *Let η , $0 < \eta < \eta_0$. Let $\mathbf{f}$ belong to $L_\eta^2(\Sigma^+)^3$ and $\mathbf{g}$ belong to $L_\eta^2(\mathbb{R}^+)^6$, let $\mathbf{h}_a$ belong to $H^{1/2}(\gamma_0)$ for each $a \in A_{\textcircled{1}}$ and $\mathbf{h}_b$ belong to $H^{-1/2}(\gamma_0)$ for each $b \in B_{\textcircled{1}}$. Then there exist $\boldsymbol{\varphi} \in H_\eta^1(\Sigma^+)^3$ and $\mathbf{Z} \in \mathcal{P}$ so that*

$$\mathcal{B}_{\textcircled{1}}(\boldsymbol{\varphi} + \mathbf{Z}) = (\mathbf{f}; \mathbf{g}; \mathbf{h}).$$

But the solution given by Lemma 4.2 is not unique. Let $\mathcal{T}_{\textcircled{1}}$ denote the space of the polynomial displacements $\mathbf{Z}$ such that there exists $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{Z}) \in H_\eta^1(\Sigma^+)^3$ satisfying

$$\mathcal{B}_{\textcircled{1}}(\mathbf{Z} + \boldsymbol{\varphi}(\mathbf{Z})) = 0.$$

Like in [7, Proposition 4.12], we can prove that

$$\dim \mathcal{T}_{\textcircled{1}} = 4.$$

Thus $\mathcal{P}$ can be split in the direct sum of two four-dimensional spaces $\mathcal{Z}_{\textcircled{1}}$ and $\mathcal{T}_{\textcircled{1}}$, and we have as corollary:

Lemma 4.3 *Let $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ be as in Lemma 4.2. Then there exist $\boldsymbol{\varphi}$ unique in $H_\eta^1(\Sigma^+)^3$ and $\mathbf{Z}$ unique in the four-dimensional space $\mathcal{Z}_{\textcircled{1}}$ so that*

$$\mathcal{B}_{\textcircled{1}}(\boldsymbol{\varphi} + \mathbf{Z}) = (\mathbf{f}; \mathbf{g}; \mathbf{h}).$$

At this stage, the conclusion is that we have a defect number equal to four for the solution of the sequence of the above equations (3.26) by exponentially decreasing displacements $\boldsymbol{\varphi}^k$, for each $s \in \partial\omega$. But four traces on $\partial\omega$ are still available, allowing to modify $\mathbf{h}^k$.

Indeed, according to the principle of ‘matching asymptotics’, the behavior at infinity of the profiles is transformed into a function of the primitive variable x (which is a Kirchhoff-Love displacement). But this is only philosophy. We prove in the following sections that for each situation $\textcircled{i}$, the expansion (2.1)-(2.2) is valid in the sense of asymptotic developments.

4.2 The operators acting on profiles

We can immediately see that the operators $\mathcal{B}_{\textcircled{1}}$ act separately on the couple of components (φ_t, φ_3) that we denote $\boldsymbol{\varphi}_{\textcircled{1}}$, and on φ_s . On $\boldsymbol{\varphi}_{\textcircled{1}}$ acts an elasticity operator with the Lamé constants λ and μ , and on φ_s a Laplace operator.

The interior elasticity operator in Σ^+ is

$$\boldsymbol{\varphi}_{\textcircled{1}} \longmapsto \mathbf{f}_{\textcircled{1}} = \mu(\partial_{tt} + \partial_{33}) \begin{pmatrix} \varphi_t \\ \varphi_3 \end{pmatrix} + (\lambda + \mu) \begin{pmatrix} \partial_t \\ \partial_3 \end{pmatrix} (\partial_t \varphi_t + \partial_3 \varphi_3), \quad (4.1)$$

its horizontal boundary conditions $\mathcal{G}^{(0)}$ (3.13) on $\gamma_{\pm}$ are

$$\varphi_{\natural} \longmapsto \mathbf{g}_{\natural} = \begin{pmatrix} \mu(\partial_3 \varphi_t + \partial_t \varphi_3) \\ (\lambda + 2\mu)\partial_3 \varphi_3 + \lambda \partial_t \varphi_t \end{pmatrix} \quad (4.2)$$

and the lateral boundary conditions are either Dirichlet's or Neumann's acting on the traction $\mathbf{T}_{\natural}^{(0)} = (T_t^{(0)}, T_3^{(0)})$, cf (3.28).

Let us introduce now the two elasticity operators used in Part I (we will use in Part II similar operators E_{Mix2} and E_{Free})

- $E_{\text{Dir}} : \varphi_{\natural} \mapsto (\mathbf{f}_{\natural}; \mathbf{g}_{\natural}; \mathbf{h}_{\natural})$ with $\mathbf{f}_{\natural}$ defined in (4.1), $\mathbf{g}_{\natural}$ defined in (4.2) and $\mathbf{h}_{\natural}$ the trace of $\varphi_{\natural}$ on γ_0 ,
- $E_{\text{Mix1}} : \varphi_{\natural} \mapsto (\mathbf{f}_{\natural}; \mathbf{g}_{\natural}; \mathbf{h}_{\natural})$ with $\mathbf{f}_{\natural}$ defined in (4.1), $\mathbf{g}_{\natural}$ defined in (4.2) and $\mathbf{h}_{\natural}$ the trace on γ_0

$$\mathbf{h}_{\natural} = \left(T_t^{(0)}(\varphi_{\natural}), \varphi_3 \right) \Big|_{\gamma_0},$$

whereas the Laplace operators are defined as:

- $L_{\text{Dir}} : \varphi_s \mapsto (\mathbf{f}_s; \mathbf{g}_s; \mathbf{h}_s)$ with $\mathbf{f}_s = \mu \Delta \varphi_s$, $\mathbf{g}_s = \mu \partial_3 \varphi_s$ and $\mathbf{h}_s = \varphi_s$ on γ_0 ,
- $L_{\text{Neu}} : \varphi_s \mapsto (\mathbf{f}_s; \mathbf{g}_s; \mathbf{h}_s)$ with $\mathbf{f}_s = \mu \Delta \varphi_s$, $\mathbf{g}_s = \mu \partial_3 \varphi_s$ and $\mathbf{h}_s = \mu \partial_t \varphi_s$ on γ_0 .

Then we have the splittings:

$$\begin{aligned} \mathcal{B}_{(1)} &= E_{\text{Dir}} \oplus L_{\text{Dir}}, & \mathcal{B}_{(2)} &= E_{\text{Dir}} \oplus L_{\text{Neu}}, \\ \mathcal{B}_{(3)} &= E_{\text{Mix1}} \oplus L_{\text{Dir}}, & \mathcal{B}_{(4)} &= E_{\text{Mix1}} \oplus L_{\text{Neu}}. \end{aligned} \quad (4.3)$$

4.3 The Laplacian on the half-strip

The Neumann problem on the full strip Σ has a polynomial kernel of dimension two generated by 1 and t , corresponding to the elements $\mathbf{Z}^{[2]}$ and $\mathbf{Z}^{[6]}$ of the space $\mathcal{P}$ introduced at the beginning of the section.

(i) OPERATOR L_{Dir} . The polynomial kernel of this problem is the function t and there holds:

Proposition 4.4 *For $\eta > 0$, let $f \in L_{\eta}^2(\Sigma^+)$, $g^{\pm} \in L_{\eta}^2(\mathbb{R}^+)^2$ and $h \in H^{1/2}(\gamma_0)$. If moreover $\eta < \pi/2$, then the problem*

$$L_{\text{Dir}}(\psi) = (f; g^{\pm}; h)$$

has a unique solution $\psi = \varphi + \delta$ in $H_{\eta}^1(\Sigma^+) \oplus \text{span}\{1\}$ with $\varphi \in H_{\eta}^1(\Sigma^+)$ and

$$\delta = \frac{1}{2\mu} \left(- \int_{\Sigma^+} t f(t, x_3) dt dx_3 + \int_{\mathbb{R}^+} t (g^+(t) - g^-(t)) dt + \mu \int_{-1}^{+1} h(x_3) dx_3 \right). \quad (4.4)$$

The proof of formula (4.4) relies on integration by parts of $t \Delta(\varphi + \delta)$ on rectangles $\Sigma_L = (0, L) \times (-1, 1)$ with $L \rightarrow +\infty$.

(ii) OPERATOR L_{Neu} . The polynomial kernel of this problem is the function 1 and there holds similarly:

Proposition 4.5 *For $\eta > 0$, let $f \in L^2_\eta(\Sigma^+)$, $g^\pm \in L^2_\eta(\mathbb{R}^+)^2$ and $h \in H^{-1/2}(\gamma_0)$. If moreover $\eta < \pi/2$, then the problem*

$$L_{\text{Neu}}(\psi) = (f; g^\pm; h)$$

has a unique solution $\psi = \varphi + \delta t$ in $H^1_\eta(\Sigma^+) \oplus \text{span}\{t\}$ with $\varphi \in H^1_\eta(\Sigma^+)$ and

$$\delta = \frac{1}{2\mu} \left(\int_{\Sigma^+} f(t, x_3) dt dx_3 - \int_{\mathbb{R}^+} (g^+(t) - g^-(t)) dt + \int_{-1}^{+1} h(x_3) dx_3 \right). \quad (4.5)$$

4.4 Elasticity on the half-strip

The problem (4.1)-(4.2) on the full strip has a polynomial kernel of dimension six generated by $\mathbf{Z}^{[1]}_{\mathfrak{h}}, \mathbf{Z}^{[3]}_{\mathfrak{h}}, \mathbf{Z}^{[4]}_{\mathfrak{h}}, \mathbf{Z}^{[5]}_{\mathfrak{h}}, \mathbf{Z}^{[7]}_{\mathfrak{h}}, \mathbf{Z}^{[8]}_{\mathfrak{h}}$, where the two components of $\mathbf{Z}^{[j]}_{\mathfrak{h}}$ are the first and third ones of $\mathbf{Z}^{[j]}$. In particular

$$\mathbf{Z}^{[1]}_{\mathfrak{h}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{Z}^{[3]}_{\mathfrak{h}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{Z}^{[4]}_{\mathfrak{h}} = \begin{pmatrix} -x_3 \\ t \end{pmatrix}$$

are the 2D rigid motions.

(i) OPERATOR E_{Dir} . From [7, Proposition 4.12], we obtain that

Proposition 4.6 *For $\eta > 0$, let $\mathbf{f}_{\mathfrak{h}} \in L^2_\eta(\Sigma^+)^2$, $\mathbf{g}^\pm_{\mathfrak{h}} \in L^2_\eta(\mathbb{R}^+)^4$ and $\mathbf{h}_{\mathfrak{h}} \in H^{1/2}(\gamma_0)^2$. If moreover $\eta < \eta_0$, then the problem*

$$E_{\text{Dir}}(\psi) = (\mathbf{f}_{\mathfrak{h}}; \mathbf{g}^\pm_{\mathfrak{h}}; \mathbf{h}_{\mathfrak{h}})$$

has a unique solution in $H^1_\eta(\Sigma^+)^2 \oplus \text{span}\{\mathbf{Z}^{[1]}_{\mathfrak{h}}, \mathbf{Z}^{[3]}_{\mathfrak{h}}, \mathbf{Z}^{[4]}_{\mathfrak{h}}\}$.

(ii) OTHER OPERATORS. Concerning the other operators E_{Mix1} , E_{Mix2} and E_{Free} , and in contrast to the case of E_{Dir} , they have a polynomial kernel generated by some of the $\mathbf{Z}^{[j]}_{\mathfrak{h}}$. Relying on the following duality relations (4.7) satisfied by the $\mathbf{Z}^{[j]}$, formulas for the coefficients in the asymptotics at infinity of the solutions can be obtained from integrations by parts. We state the corresponding result for E_{Mix1} , leaving the others for Part II.

Lemma 4.7 *Let $\mathbf{T}^{(0)}$ denote the lateral inward traction operator $(T_t^{(0)}, T_s^{(0)}, T_3^{(0)})$, see (3.4). With σ the permutation*

$$\sigma(1) = 5, \quad \sigma(2) = 6, \quad \sigma(3) = 8, \quad \sigma(4) = 7,$$

$$\sigma(5) = 1, \quad \sigma(6) = 2, \quad \sigma(7) = 4, \quad \sigma(8) = 3,$$

the anti-symmetrized flux (see also (5.17)), which can be defined for any $L \in \mathbb{R}$ by

$$\underline{\Phi}(\mathbf{Z}^{[i]}, \mathbf{Z}^{[j]}) := \int_{-1}^{+1} (\mathbf{T}^{(0)}(\mathbf{Z}^{[i]}) \cdot \mathbf{Z}^{[j]} - \mathbf{T}^{(0)}(\mathbf{Z}^{[j]}) \cdot \mathbf{Z}^{[i]})(L, x_3) dx_3 \quad (4.6)$$

is independent of L , compare [7, Lemma 3.1], and satisfies, for $i, j \in \{1, \dots, 8\}$

$$\Phi(\mathbf{Z}^{[i]}, \mathbf{Z}^{[j]}) = \bar{\gamma}_i \delta_{j\sigma(i)}, \quad (4.7)$$

with $\bar{\gamma}_i$ a non zero real number.

For $i = 2, 6$ we find again the simple relations on which rely Propositions 4.4 and 4.5. For the remaining values of i , the relations (4.7) apply to the bi-dimensional displacements $\mathbf{Z}_{\mathfrak{h}}^{[i]}$.

Proposition 4.8 *For $\eta > 0$, let $\mathbf{f}_{\mathfrak{h}} \in L_{\eta}^2(\Sigma^+)^2$, $\mathbf{g}_{\mathfrak{h}}^{\pm} \in L_{\eta}^2(\mathbb{R}^+)^4$, $\mathbf{h}_t \in H^{-1/2}(\gamma_0)$ and $\mathbf{h}_3 \in H^{1/2}(\gamma_0)$. If moreover $\eta < \eta_0$, then the problem*

$$E_{\text{Mix1}}(\psi) = (\mathbf{f}_{\mathfrak{h}}; \mathbf{g}_{\mathfrak{h}}^{\pm}; \mathbf{h}_{\mathfrak{h}})$$

has a unique solution in $H_{\eta}^1(\Sigma^+)^2 \oplus \text{span}\{\mathbf{Z}_{\mathfrak{h}}^{[3]}, \mathbf{Z}_{\mathfrak{h}}^{[5]}, \mathbf{Z}_{\mathfrak{h}}^{[7]}\}$. Moreover

$$\psi = \varphi + \delta_3 \mathbf{Z}_{\mathfrak{h}}^{[3]} + \delta_5 \mathbf{Z}_{\mathfrak{h}}^{[5]} + \delta_7 \mathbf{Z}_{\mathfrak{h}}^{[7]} \quad \text{with} \quad \varphi \in H_{\eta}^1(\Sigma^+)^2$$

and for $j = 1, 4, 8$

$$\begin{aligned} \delta_{\sigma(j)} = \frac{1}{\bar{\gamma}_{\sigma(j)}} & \left(\int_{\Sigma^+} \mathbf{f}_{\mathfrak{h}} \cdot \mathbf{Z}_{\mathfrak{h}}^{[j]} dt dx_3 - \int_{\mathbb{R}^+} \left(\mathbf{g}^+ \cdot \mathbf{Z}_{\mathfrak{h}}^{[j]} \Big|_{\gamma^+} - \mathbf{g}^- \cdot \mathbf{Z}_{\mathfrak{h}}^{[j]} \Big|_{\gamma^-} \right) dt \right. \\ & \left. + \int_{-1}^{+1} \mathbf{h}_t \mathbf{Z}_t^{[j]} - \mathbf{h}_3 T_3^{(0)}(\mathbf{Z}_{\mathfrak{h}}^{[j]}) dx_3 \right), \end{aligned}$$

namely

$$\bar{\gamma}_5 \delta_5 = \int_{\Sigma^+} f_t - \int_{\mathbb{R}^+} (g_t^+ - g_t^-) + \int_{-1}^{+1} \mathbf{h}_t, \quad (4.8a)$$

$$\bar{\gamma}_7 \delta_7 = \int_{\Sigma^+} (-x_3 f_t + t f_3) + \int_{\mathbb{R}^+} (g_t^+ + g_t^- - t(g_3^+ - g_3^-)) - \int_{-1}^{+1} x_3 \mathbf{h}_t, \quad (4.8b)$$

$$\bar{\gamma}_3 \delta_3 = \int_{\Sigma^+} \mathbf{f}_{\mathfrak{h}} \cdot \mathbf{Z}_{\mathfrak{h}}^{[8]} - \int_{\mathbb{R}^+} \left(\mathbf{g}^+ \cdot \mathbf{Z}_{\mathfrak{h}}^{[8]} \Big|_{\gamma^+} - \mathbf{g}^- \cdot \mathbf{Z}_{\mathfrak{h}}^{[8]} \Big|_{\gamma^-} \right) + 6 \int_{-1}^{+1} \bar{p}_3 \mathbf{h}_t - \mu(\bar{p}_2 + \bar{p}_3') \mathbf{h}_3. \quad (4.8c)$$

5 HARD CLAMPED

The main results concerning the case ① are summarized in the following

Theorem 5.1 *The expansion (2.1)-(2.2) holds for the solution $\mathbf{u}(\varepsilon)$ of the scaled variant of problem (1.1) with lateral condition ①. This expansion is uniquely determined (a change of the cut-off χ would only have an exponentially decreasing influence in ε). In particular the Kirchhoff-Love generators ζ^0 and ζ^1 satisfy the following Dirichlet boundary conditions on $\partial\omega$*

$$\begin{aligned} \zeta_n^0 &= 0, & \zeta_s^0 &= 0, & \zeta_3^0 &= 0, & \partial_n \zeta_3^0 &= 0, \\ \zeta_n^1 &= c_1^{(1)} \text{div}_* \zeta_*^0, & \zeta_s^1 &= 0, & \zeta_3^1 &= 0, & \partial_n \zeta_3^1 &= c_4^{(1)} \Delta_* \zeta_3^0, \end{aligned} \quad (5.1)$$

where $c_1^{(1)} = c_1^{(1)}(\lambda, \mu)$ and $c_4^{(1)} = c_4^{(1)}(\lambda, \mu)$ are universal coefficients with

$$\forall \lambda, \mu > 0 \quad c_4^{(1)}(\lambda, \mu) \neq 0. \quad (5.2)$$

The first boundary layer profile φ^1 satisfies $\varphi_s^1 = 0$ and

$$(\varphi_t^1, \varphi_3^1) =: \varphi_{\natural}^1 = \operatorname{div}_* \zeta_*^0(s) \bar{\varphi}_{\natural}^m + \Delta_* \zeta_3^0(s) \bar{\varphi}_{\natural}^b, \quad (5.3)$$

where $\bar{\varphi}_{\natural}^m = \bar{\varphi}_{\operatorname{Dir}, \natural}^m$ and $\bar{\varphi}_{\natural}^b = \bar{\varphi}_{\operatorname{Dir}, \natural}^b$ are typical profiles depending only on (t, x_3) .

In [6, 7] we already proved the validity of the expansion (2.1)-(2.2) in the sense of asymptotic developments, i.e. that optimal estimates hold for the remainders at any order and in any norm $H^s(\Omega)$ with $s \leq 1$ — and also for the uniform norm $L^\infty(\Omega)$.

The homogeneous Dirichlet boundary conditions satisfied by ζ^0 are also well known, cf [4, 8] for early reference. With our presentation, this is a trivial consequence of condition (3.18) for $k = 0$.

The mere existence of $\mathbf{u}_{\operatorname{KL}}^1$ is less known. We are going to prove the above formulas for the Dirichlet traces of ζ^1 and exhibit the constants $c_1^{(1)}$ and $c_4^{(1)}$, see (5.10) and (5.12), then we prove that $c_4^{(1)}$ is non-zero.

5.1 The Dirichlet traces of $\mathbf{u}_{\operatorname{KL}}^1$

At this stage, $\mathbf{u}_{\operatorname{KL}}^0$ is completely known, since ζ_*^0 is the solution of the membrane equation $L^m(\zeta_*^0) = \mathbf{R}_m^0$ with null Dirichlet conditions and ζ_3^0 is the solution of the bending equation $L^b(\zeta_3^0) = R_b^0$ with null Dirichlet conditions.

Condition (3.18) for $k = 1$ yields that $\zeta_3^1 = 0$, $\varphi_n^1 + \zeta_n^1 - x_3 \partial_n \zeta_3^1 = 0$ and $\varphi_s^1 + \zeta_s^1 - x_3 \partial_s \zeta_3^1 = 0$ on Γ_0 . Moreover condition (3.18) for $k = 2$ with (3.20c) yields that $\varphi_3^1 + \zeta_3^2 + v_3^2 = 0$ on Γ_0 .

Thus, the first profile $\varphi^1(s) : (t, x_3) \mapsto \varphi^1(t, s, x_3)$ has to solve for all $s \in \partial\omega$ — cf equation (3.26)

$$\mathcal{B}_{\textcircled{1}}(\varphi^1(s)) = (0; 0; \mathbf{h}^1(s)) \quad (5.4)$$

with the trace $\mathbf{h}^1(s)$ equal to, for $s \in \partial\omega$:

$$\mathbf{h}_n^1(s) = -(\zeta_n^1 - x_3 \partial_n \zeta_3^1)(s) \quad (5.5a)$$

$$\mathbf{h}_s^1(s) = -(\zeta_s^1 - x_3 \partial_s \zeta_3^1)(s) \quad (5.5b)$$

$$\mathbf{h}_3^1(s) = -(\zeta_3^2 + v_3^2)(s). \quad (5.5c)$$

Note that the unknowns are the profile φ^1 and the traces of ζ_n^1 , ζ_s^1 , $\partial_n \zeta_3^1$ and ζ_3^2 .

Since $\mathcal{B}_{\textcircled{1}}$ splits into the direct sum $E_{\operatorname{Dir}} \oplus L_{\operatorname{Dir}}$, see (4.3), for each $s \in \partial\omega$ (fixed now, thus omitted),

- φ_s^1 is solution of the Poisson problem

$$L_{\operatorname{Dir}}(\varphi_s^1) = (0; 0; \mathbf{h}_s^1), \quad (5.6)$$

- the couple $\varphi_{\mathfrak{h}}^1$ is solution of the elasticity system

$$E_{\text{Dir}}(\varphi_{\mathfrak{h}}^1) = (0; 0; \mathfrak{h}_{\mathfrak{h}}^1). \quad (5.7)$$

We have to find the conditions on ζ^1 so that equations (5.6) and (5.7) admit exponentially decreasing solutions.

Concerning the Poisson problem, Proposition 4.4 yields that (5.6) admits an exponentially decreasing solution if the coefficient (4.4) is zero, i.e. if $\int_{-1}^{+1} \mathfrak{h}_s^1 = 0$. With (5.5b), this yields that $\zeta_s^1 = 0$ on $\partial\omega$. Since we already found that $\zeta_3^1 = 0$ on $\partial\omega$, we obtain that $\mathfrak{h}_s^1 \equiv 0$, thus $\varphi_s^1 = 0$.

Concerning the Lamé problem, Proposition 4.6 yields a solution for (5.7) in $H_{\eta}^1(\Sigma^+)^2 \oplus \text{span}\{\mathbf{Z}_{\mathfrak{h}}^{[1]}, \mathbf{Z}_{\mathfrak{h}}^{[3]}, \mathbf{Z}_{\mathfrak{h}}^{[4]}\}$ (a space of exponentially decaying solutions plus a space of rigid motions). We first recall that, — cf (2.6)-(2.7)

$$v_3^2(x_*, x_3) = \bar{p}_1(x_3) \text{div}_* \zeta_*^0(x_*) + \bar{p}_2(x_3) \Delta_* \zeta_3^0(x_*), \quad (5.8)$$

where $\bar{p}_1$ is an odd polynomial, and $\bar{p}_2$ an even polynomial.

Let $\bar{\psi}_{\mathfrak{h}}^{\text{m}}$ be the solution in $H_{\eta}^1(\Sigma^+)^2 \oplus \text{span}\{\mathbf{Z}_{\mathfrak{h}}^{[1]}, \mathbf{Z}_{\mathfrak{h}}^{[3]}, \mathbf{Z}_{\mathfrak{h}}^{[4]}\}$ of

$$E_{\text{Dir}}(\bar{\psi}_{\mathfrak{h}}^{\text{m}}) = (0; 0; 0, -\bar{p}_1). \quad (5.9)$$

Since the right hand side of (5.9) has the parities of a membrane mode (the first component is even and the second odd with respect to x_3), the symmetries of the isotropic elasticity system yield that $\bar{\psi}_t^{\text{m}}$ is even and $\bar{\psi}_3^{\text{m}}$ odd. Thus the asymptotic behavior as $t \rightarrow \infty$ has the same parities: only $\mathbf{Z}_{\mathfrak{h}}^{[1]}$ is convenient. Hence there exists a unique coefficient $c_1^{(1)}$ such that $\bar{\psi}_{\mathfrak{h}}^{\text{m}}$ splits into

$$\bar{\psi}_{\mathfrak{h}}^{\text{m}} = \bar{\varphi}_{\mathfrak{h}}^{\text{m}} + c_1^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[1]} \quad \text{with } \bar{\varphi}_{\mathfrak{h}}^{\text{m}} \text{ exponentially decreasing.} \quad (5.10)$$

Similarly, let $\bar{\psi}_{\mathfrak{h}}^{\text{b}}$ be the solution in $H_{\eta}^1(\Sigma^+)^2 \oplus \text{span}\{\mathbf{Z}_{\mathfrak{h}}^{[1]}, \mathbf{Z}_{\mathfrak{h}}^{[3]}, \mathbf{Z}_{\mathfrak{h}}^{[4]}\}$ of

$$E_{\text{Dir}}(\bar{\psi}_{\mathfrak{h}}^{\text{b}}) = (0; 0; 0, -\bar{p}_2). \quad (5.11)$$

Since the right hand side of (5.11) has the parities of a bending mode, the symmetries of the problem yield that $\bar{\psi}_t^{\text{b}}$ is odd and $\bar{\psi}_3^{\text{b}}$ even with respect to x_3 . Thus only $\mathbf{Z}_{\mathfrak{h}}^{[3]}$ and $\mathbf{Z}_{\mathfrak{h}}^{[4]}$ are present in the asymptotics at infinity of $\bar{\psi}_{\mathfrak{h}}^{\text{b}}$. Hence there exist unique coefficients $c_3^{(1)}$ and $c_4^{(1)}$ such that $\bar{\psi}_{\mathfrak{h}}^{\text{b}}$ splits into

$$\bar{\psi}_{\mathfrak{h}}^{\text{b}} = \bar{\varphi}_{\mathfrak{h}}^{\text{b}} + c_3^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[3]} + c_4^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[4]} \quad \text{with } \bar{\varphi}_{\mathfrak{h}}^{\text{b}} \text{ exponentially decreasing.} \quad (5.12)$$

Then $\psi_{\mathfrak{h}}^1$ defined as

$$\psi_{\mathfrak{h}}^1(t, s, x_3) = \text{div}_* \zeta_*^0(s) \bar{\psi}_{\mathfrak{h}}^{\text{m}}(t, x_3) + \Delta_* \zeta_3^0(s) \bar{\psi}_{\mathfrak{h}}^{\text{b}}(t, x_3)$$

is solution for each $s \in \partial\omega$ of — cf (5.8), (5.9) and (5.11):

$$E_{\text{Dir}}(\boldsymbol{\psi}_{\natural}^1) = (0; 0; 0, -v_3^2). \quad (5.13)$$

Thus, if we have for each $s \in \partial\omega$, cf (5.10) and (5.12)

$$\begin{pmatrix} \zeta_n^1(s) - x_3 \partial_n \zeta_3^1(s) \\ \zeta_3^2(s) \end{pmatrix} = \text{div}_* \boldsymbol{\zeta}_*^0(s) c_1^{(1)} \mathbf{Z}_{\natural}^{[1]} \Big|_{\gamma_0} + \Delta_* \zeta_3^0(s) (c_3^{(1)} \mathbf{Z}_{\natural}^{[3]} + c_4^{(1)} \mathbf{Z}_{\natural}^{[4]}) \Big|_{\gamma_0}$$

i.e.

$$\begin{pmatrix} \zeta_n^1(s) - x_3 \partial_n \zeta_3^1(s) \\ \zeta_3^2(s) \end{pmatrix} = \begin{pmatrix} \text{div}_* \boldsymbol{\zeta}_*^0(s) c_1^{(1)} - x_3 \Delta_* \zeta_3^0(s) c_4^{(1)} \\ \Delta_* \zeta_3^0(s) c_3^{(1)} \end{pmatrix} \quad (5.14)$$

then $\boldsymbol{\varphi}_{\natural}^1$ defined as

$$\boldsymbol{\varphi}_{\natural}^1(t, s, x_3) = \text{div}_* \boldsymbol{\zeta}_*^0(s) \bar{\boldsymbol{\varphi}}_{\natural}^m(t, x_3) + \Delta_* \zeta_3^0(s) \bar{\boldsymbol{\varphi}}_{\natural}^b(t, x_3) \quad (5.15)$$

is solution of $E_{\text{Dir}}(\boldsymbol{\varphi}_{\natural}^1(s)) = (0; 0; \mathbf{h}_{\natural}^1(s))$, see (5.4), (5.5). We see that the first line of (5.14) is nothing else than (5.1).

5.2 The non-zero coefficient $c_4^{(1)}$

Let us prove now that the coefficient $c_4^{(1)}$ is not zero. Let us denote by $\mathbf{Z}_{\natural}$ the polynomial displacement

$$\mathbf{Z}_{\natural} = \begin{pmatrix} -tx_3 \\ t^2 \\ \frac{t^2}{2} + \bar{p}_2 \end{pmatrix}.$$

Indeed, $\mathbf{Z}_{\natural}$ is issued from $\mathbf{Z}^{[7]}$ and satisfies:

$$E_{\text{Dir}}(\mathbf{Z}_{\natural}) = (0; 0; 0, \bar{p}_2). \quad (5.16)$$

So, (5.16) joined with (5.11)-(5.12) yields that

$$\mathbf{K} := \mathbf{Z}_{\natural} + \bar{\boldsymbol{\varphi}}_{\natural}^b + c_3^{(1)} \mathbf{Z}_{\natural}^{[3]} + c_4^{(1)} \mathbf{Z}_{\natural}^{[4]} \in \ker E_{\text{Dir}}.$$

The proof proceeds by computation about the ‘flux’, see also (4.6)

$$\Phi_{t=t_0}(\mathbf{u} \mid \mathbf{v}) := \int_{-1}^{+1} \mathbf{T}^{(0)}(\mathbf{u})(t_0, x_3) \cdot \mathbf{v}(t_0, x_3) dx_3 \quad (5.17)$$

with $\mathbf{T}^{(0)}$ denoting the lateral inward traction operator $(T_t^{(0)}, T_s^{(0)}, T_3^{(0)})$, cf (3.4). We have:

$$\mathbf{T}^{(0)}(\mathbf{Z}_{\natural}) = \begin{pmatrix} -4 \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} x_3 \\ 0 \end{pmatrix}.$$

Thus

$$\Phi_{t=0}(\mathbf{Z}_{\natural} \mid c_3^{(1)} \mathbf{Z}_{\natural}^{[3]} + c_4^{(1)} \mathbf{Z}_{\natural}^{[4]}) = \frac{8}{3} \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} c_4^{(1)}. \quad (5.18)$$

We are going to prove that

$$\Phi_{t=0}(\mathbf{Z}_{\mathfrak{h}} | c_3^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[3]} + c_4^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[4]}) = \Phi_{t=0}(\mathbf{K} | c_3^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[3]} + c_4^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[4]} + \bar{\varphi}_{\mathfrak{h}}^b) \quad (5.19)$$

and that

$$\Phi_{t=0}(\mathbf{K} | c_3^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[3]} + c_4^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[4]} + \bar{\varphi}_{\mathfrak{h}}^b) > 0. \quad (5.20)$$

The fact that $c_4^{(1)} > 0$ is clearly a consequence of (5.18)-(5.20).

In order to prove (5.19) and (5.20), we abbreviate the notations by

$$c_3^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[3]} + c_4^{(1)} \mathbf{Z}_{\mathfrak{h}}^{[4]} := \mathbf{R} \quad \text{and} \quad \varphi := \bar{\varphi}_{\mathfrak{h}}^b.$$

Proof of (5.19). We want to prove that

$$\Phi_{t=0}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{R}) = \Phi_{t=0}(\mathbf{K} | \mathbf{R} + \varphi).$$

Indeed, integrating by parts on the rectangle $\Sigma_L = (0, L) \times (-1, 1)$ we obtain

$$\int_{\partial \Sigma_L} \mathbf{T}^{(0)}(\mathbf{K}) \cdot (\mathbf{R} + \varphi) - \mathbf{K} \cdot \mathbf{T}^{(0)}(\mathbf{R} + \varphi) = \int_{\Sigma_L} E(\mathbf{K}) \cdot (\mathbf{R} + \varphi) - \mathbf{K} \cdot E(\mathbf{R} + \varphi).$$

As $E\mathbf{K} = E\mathbf{Z}_{\mathfrak{h}} = 0$, the above right hand side is zero. Therefore

$$\Phi_{t=0}(\mathbf{K} | \mathbf{R} + \varphi) = \Phi_{t=L}(\mathbf{K} | \mathbf{R} + \varphi) + \int_{-1}^{+1} \mathbf{K}(L, x_3) \cdot \mathbf{T}^{(0)}(\mathbf{R} + \varphi)(L, x_3) dx_3.$$

Since $\mathbf{T}^{(0)}(\mathbf{R}) = 0$ ($\mathbf{R}$ is a rigid displacement) and since φ is exponentially decreasing, we deduce from the identity above that, for all $0 < \eta < \eta_0$

$$\Phi_{t=0}(\mathbf{K} | \mathbf{R} + \varphi) = \Phi_{t=L}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{R}) + \mathcal{O}(e^{-\eta L}).$$

But for all L , we have the conservation of the flux against rigid displacements

$$\Phi_{t=L}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{R}) = \Phi_{t=0}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{R}),$$

whence the result. ■

Proof of (5.20). We want to prove that

$$\Phi_{t=0}(\mathbf{K} | \mathbf{R} + \varphi) > 0.$$

To see it, notice that, since $\mathbf{Z}_{\mathfrak{h}}|_{t=0} = -(\mathbf{R} + \varphi)|_{t=0}$ and since we easily check the equality $\Phi_{t=0}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{Z}_{\mathfrak{h}}) = 0$, we have

$$\begin{aligned} \Phi_{t=0}(\mathbf{K} | \mathbf{R} + \varphi) &= \Phi_{t=0}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{R} + \varphi) + \Phi_{t=0}(\mathbf{R} + \varphi | \mathbf{R} + \varphi) \\ &= -\Phi_{t=0}(\mathbf{Z}_{\mathfrak{h}} | \mathbf{Z}_{\mathfrak{h}}) + \Phi_{t=0}(\varphi | \mathbf{R} + \varphi) \\ &= \Phi_{t=L}(\varphi | \mathbf{R} + \varphi) + \int_{\Sigma_L} A e(\partial_t, \partial_3)(\varphi) : e(\partial_t, \partial_3)(\mathbf{R} + \varphi) \\ &= \int_{\Sigma_L} A e(\partial_t, \partial_3)(\varphi) : e(\partial_t, \partial_3)(\varphi) + \mathcal{O}(e^{-\eta L}). \end{aligned}$$

Since $\mathbf{Z}_{\mathfrak{h}} + \mathbf{R}$ is clearly not zero on $\{t = 0\}$, then $\varphi \not\equiv 0$. The result follows from the positivity of the elasticity matrix A . ■

6 SOFT CLAMPED

We have now to take care of the space $\mathcal{R}_{\textcircled{2}}$, cf (1.2)-(1.3): let us recall that $\mathcal{R}_{\textcircled{2}}$ is the space of rigid motions $\mathbf{v}$ satisfying the soft clamped plate conditions, i.e. v_n and $v_3 = 0$ on the lateral boundary Γ_0 . If the mean surface ω is not a disk or an annulus, $\mathcal{R}_{\textcircled{2}}$ is reduced to $\{0\}$. If ω is a disk or an annulus, that we may suppose centered in 0 , $\mathcal{R}_{\textcircled{2}}$ is one-dimensional, generated by the in-plane rotation $(x_2, -x_1, 0)$ and the orthogonality condition (1.3) ensuring uniqueness can be transcribed in Ω into $\int_{\Omega} \mathbf{u}_*(\varepsilon) \times x_* = 0$, with x_* the vector (x_1, x_2) .

Theorem 6.1 *The expansion (2.1)-(2.2) holds for the solution $\mathbf{u}(\varepsilon)$ of the scaled variant of problem (1.1) with lateral condition $\textcircled{2}$, compatibility condition (1.2) and orthogonality condition (1.3). The Kirchhoff-Love generators ζ_*^k and ζ_3^k are solutions of membrane and bending problems with interior data $\mathbf{R}_m^k$ and R_b^k , cf Table 6, and boundary conditions prescribing ζ_n^k , $T_s^m(\zeta_*^k)$, and ζ_3^k , $\partial_n \zeta_3^k$ respectively. The Kirchhoff-Love generators ζ^0 and ζ^1 satisfy the following boundary conditions on $\partial\omega$*

$$\begin{aligned} \zeta_n^0 &= 0, & T_s^m(\zeta_*^0) &= 0, & \zeta_3^0 &= 0, & \partial_n \zeta_3^0 &= 0, \\ \zeta_n^1 &= c_1^{(2)} \operatorname{div}_* \zeta_*^0, & T_s^m(\zeta_*^1) &= c_2^{(2)} \partial_s \operatorname{div}_* \zeta_*^0, & \zeta_3^1 &= 0, & \partial_n \zeta_3^1 &= c_4^{(2)} \Delta_* \zeta_3^0, \end{aligned} \quad (6.1)$$

where $c_1^{(2)}$ and $c_4^{(2)}$ are respectively equal to the coefficients $c_1^{(1)}$ and $c_4^{(1)}$ introduced in Theorem 5.1 and $c_2^{(2)}$ is another universal coefficient. The first boundary layer profile satisfies $\varphi_s^1 = 0$ and $\varphi_{\mathfrak{t}}^1$ is given by formula (5.3) like in the hard clamped plate situation.

If ω is a disk or an annulus with center 0 the compatibility condition

$$\int_{\omega} \mathbf{R}_m^k \times x_* + \int_{\partial\omega} |x_*| T_s^m(\zeta_*^k) = 0$$

holds for each k as well as the orthogonality condition $\int_{\omega} \zeta_*^k \times x_* = 0$.

The behavior of the boundary layer terms is very similar to the hard clamped case because the boundary conditions involving the components $\varphi_{\mathfrak{t}}$ are Dirichlet's as in $\textcircled{1}$, the only change concerns the lateral component φ_s , which is uncoupled from the previous ones, and subject now to lateral Neumann conditions instead of Dirichlet's.

6.1 The traces of $\mathbf{u}_{\text{KL}}^0$

Solving recursively equations (3.16)-(3.17), (3.18) and (3.21), we find first the Dirichlet traces at the order zero: D_n^0 and $D_3^0 = 0$, i.e. $u_{\text{KL},n}^0$ and $u_{\text{KL},3}^0$ are zero on Γ_0 , whence $\zeta_n^0 - x_3 \partial_n \zeta_3^0$ and ζ_3^0 are zero on $\partial\omega$. Thus, the Dirichlet conditions concerning ζ^0 are obtained.

As already remarked in (3.24), the first terms T_s^0 and T_s^1 are always zero. Next, condition $T_s^2 = 0$ yields, cf (3.25b)

$$T_s^{(0)}(\varphi^1) = -T_s^m(\zeta_*^0) + 2\mu x_3 (\partial_n + \frac{1}{R}) \partial_s \zeta_3^0.$$

Taking account of the already known Dirichlet conditions for ζ_3^0 , we obtain that φ_s^1 solves the Laplace Neumann problem on the half-strip:

$$L_{\text{Neu}}(\varphi_s^1) = (0; 0; -T_s^m(\zeta_*^0)). \quad (6.2)$$

Since, for each fixed $s \in \partial\omega$, $T_s^m(\zeta_*^0)$ is a constant, Proposition 4.5 yields that the only exponentially decreasing solution is $\varphi_s^1 \equiv 0$ obtained with $T_s^m(\zeta_*^0) = 0$ on $\partial\omega$.

Then ζ_*^0 satisfies boundary conditions (6.1). In the special case when ω is a disk or an annulus, the compatibility condition (1.2) yields that the right hand side $\mathbf{R}_m^0$ of the membrane equation is orthogonal to $(x_2, -x_1)$, whence the existence of ζ_*^0 — the uniqueness being achieved in a coherent way as indicated in Theorem 6.1.

6.2 The traces of u_{KL}^1

The equations (3.18) for $k = 1$ and for $k = 2$ yield the same condition as in case ① for the trace of ζ_3^1 which must vanish, and the same equations (5.7) linking the couple $\varphi_{\natural}^1$ and the traces of ζ_n^1 , $\partial_n \zeta_3^1$, ζ_3^2 . Thus, the result concerning these traces is the same for the hard and soft clamped situations.

Concerning the tangential component, the condition $T_s^3 = 0$ yields, cf (3.25b)

$$T_s^{(0)}(\varphi^2) = -T_s^m(\zeta_*^1) + 2\mu x_3 (\partial_n + \frac{1}{R}) \partial_s \zeta_3^1 - T_s^{(1)}(\varphi^1).$$

Taking into account the already known trace condition $\zeta_3^1 = 0$, equation (3.26) leads to the following Neumann problem for the lateral part φ_s^2

$$L_{\text{Neu}}(\varphi_s^2) = \left(-(\mathcal{B}^{(1)}\varphi^1)_s; -(\mathcal{G}^{(1)}\varphi^1)_s; -T_s^m(\zeta_*^1) + 2\mu x_3 \partial_{sn} \zeta_3^1 - T_s^{(1)}(\varphi^1) \right). \quad (6.3)$$

Proposition 4.5 yields that φ_s^2 is exponentially decreasing if and only if

$$\begin{aligned} T_s^m(\zeta_*^1) = & -\frac{1}{2} \left(\int_{\Sigma_+} (\mathcal{B}^{(1)}\varphi^1)_s(t, x_3) dt dx_3 \right. \\ & - \int_{\mathbb{R}_+} \left((\mathcal{G}^{(1)}\varphi^1)_s(t, 1) - (\mathcal{G}^{(1)}\varphi^1)_s(t, -1) \right) dt \\ & \left. + \int_{-1}^{+1} T_s^{(1)}(\varphi^1)(0, x_3) - 2\mu x_3 \partial_{sn} \zeta_3^1(0) dx_3 \right). \end{aligned} \quad (6.4)$$

Since $\varphi_s^1 = 0$, the terms involved in (6.4) reduce to

$$(\mathcal{B}^{(1)}\varphi^1)_s = (\lambda + \mu) \partial_s (\partial_t \varphi_t^1 + \partial_3 \varphi_3^1), \quad (\mathcal{G}^{(1)}\varphi^1)_s = \mu \partial_s \varphi_3^1, \quad T_s^{(1)}(\varphi^1) = \mu \partial_s \varphi_t^1.$$

Since only the even terms in x_3 contribute to the integrals in (6.4) we see that we have only to take into consideration the (even, odd) part of $\varphi_{\natural}^1$, which is equal to, cf (5.15)

$$\text{div}_* \zeta_*^0(s) \bar{\varphi}_{\natural}^m(t, x_3).$$

Thus $T_s^m(\zeta_*^1) = c_2^{(2)} \partial_s \operatorname{div}_* \zeta_*^0$, with

$$\begin{aligned} -2c_2^{(2)} &= (\lambda + \mu) \int_{\Sigma_+} (\partial_t \bar{\varphi}_t^m + \partial_3 \bar{\varphi}_3^m) dt dx_3 \\ &\quad - \mu \int_{\mathbb{R}_+} (\bar{\varphi}_3^m(t, 1) - \bar{\varphi}_3^m(t, -1)) dt \\ &\quad + \mu \int_{-1}^{+1} \bar{\varphi}_t^m(0, x_3) dx_3. \end{aligned} \quad (6.5)$$

Formulas (6.1) are completely proved.

Thus, ζ^1 is the unique solution of the membrane and bending equations with zero interior right hand sides and the boundary conditions of (6.1), except when ω is a disk or an annulus, where we check the orthogonality condition $\int_{\partial\omega} T_s^m(\zeta_*^1) ds = 0$ due to the identity $\int_{\partial\omega} \partial_s \operatorname{div}_* \zeta_*^0 ds = 0$.

6.3 Recursivity

Recursion hypothesis. Let us assume that ζ^k is determined for $k \leq j-2$, $\zeta_n^k|_{\partial\omega}$ and $\partial_n \zeta_3^k|_{\partial\omega}$ for $k \leq j-1$ and $\zeta_3^k|_{\partial\omega}$ for $k \leq j$. And finally that φ^k is determined for $k \leq j-1$ such that (3.16)-(3.17) is satisfied for $k \leq j-1$, $D_n^k = 0$ for $k \leq j-1$, $D_3^k = 0$ for $k \leq j$, cf (3.18) and $T_s^k = 0$ for $k \leq j$, cf (3.21).

Therefore, by the formulas of Table 6, $\mathbf{v}^k$ and $\mathbf{R}_m^k$ are known for $k \leq j$.

The condition $T_s^{j+1} = 0$ yields that the tangential part φ_s^j of problem (3.26) solves

$$L_{\text{Neu}}(\varphi_s^j) = (\mathbf{f}_s^j; \mathbf{g}_s^j; \mathbf{h}_s^j), \quad (6.6)$$

where $\mathbf{f}_s^j$ and $\mathbf{g}_s^j$ are given by (3.27) and $\mathbf{h}_s^j = T_s^m(\zeta_*^{j-1}) + \mathbf{l}_s^j$ where

$$\mathbf{l}_s^j = 2\mu x_3 (\partial_n + \frac{1}{R}) \partial_s \zeta_3^{j-1} - \mu (\partial_s \mathbf{v}_n^{j-1} + \partial_n \mathbf{v}_s^{j-1} + \frac{2}{R} \mathbf{v}_s^{j-1}) - T_s^{(1)}(\varphi^{j-1}). \quad (6.7)$$

As $\mathbf{f}_s^j$, $\mathbf{g}_s^j$ and $\mathbf{l}_s^j$ are known, Proposition 4.5 yields the value of the trace of $T_s^m(\zeta_*^{j-1})$ so that the coefficient (4.5) is zero, and thus φ_s^j is exponentially decreasing.

Then ζ_*^{j-1} is determined, so is R_b^{j-1} , thus ζ^{j-1} is known. Therefore $\mathbf{v}^{j+1}$ is known and conditions $D_n^j = 0$ and $D_3^{j+1} = 0$ yield that φ_b^j solves

$$E_{\text{Dir}}(\varphi_b^j) = (\mathbf{f}_b^j; \mathbf{g}_b^j; -(\zeta_n^j - x_3 \partial_n \zeta_3^j + v_n^j), -(\zeta_3^{j+1} + v_3^{j+1})). \quad (6.8)$$

As already explained in §5.1, there exist unique traces $\zeta_n^j|_{\partial\omega}$, $\partial_n \zeta_3^j|_{\partial\omega}$ and $\zeta_3^{j+1}|_{\partial\omega}$ such that problem (6.8) has a unique exponentially decreasing solution. The recurrence is now completed.

When ω is a disk or an annulus, we can check that the orthogonality conditions ensuring the existence of ζ_*^j are automatically fulfilled: indeed $\int_{\omega} \mathbf{R}_m^j \times x_* = 0$ holds because of the special form of $\mathbf{R}_m^j$, cf Def. 2.7 and Table 6, plus an integration by parts; moreover $\int_{\partial\omega} T_s^m(\zeta_*^j) = 0$ due to a similar argument as in the case $j = 1$, plus the fact that the transverse mean values of the $\mathbf{v}^k$ are zero, as well as the transverse mean values of the φ_s^k . The orthogonality condition $\int_{\omega} \zeta_*^j \times x_* = 0$ is then a consequence of (1.3), since the boundary layer terms do not contribute to the integral $\int_{\Omega} \mathbf{u}_*(\varepsilon) \times x_*$.

7 HARD SIMPLY SUPPORTED

Here the space of rigid motions $\mathcal{R}_{\textcircled{3}}$ is reduced to $\{0\}$.

Theorem 7.1 *The expansion (2.1)-(2.2) holds for the solution $\mathbf{u}(\varepsilon)$ of the scaled variant of problem (1.1) with lateral condition $\textcircled{3}$. The Kirchhoff-Love generators ζ_*^k and ζ_3^k are solutions of membrane and bending problems with interior data $\mathbf{R}_m^k$ and R_b^k , and boundary conditions prescribing $T_n^m(\zeta_*^k)$, ζ_s^k , and ζ_3^k , $M_n(\zeta_3^k)$ respectively. Namely, ζ^0 and ζ^1 satisfy the following boundary conditions on $\partial\omega$*

$$\begin{aligned} T_n^m(\zeta_*^0) &= 0, & \zeta_s^0 &= 0, & \zeta_3^0 &= 0, & M_n(\zeta_3^0) &= 0, \\ T_n^m(\zeta_*^1) &= c_1^{\textcircled{3}} \kappa^2 \zeta_n^0, & \zeta_s^1 &= 0, & \zeta_3^1 &= 0, & M_n(\zeta_3^1) &= c_4^{\textcircled{3}} \kappa^2 \partial_n \zeta_3^0, \end{aligned} \quad (7.1)$$

where $c_1^{\textcircled{3}}$ and $c_4^{\textcircled{3}}$ are coefficients only depending on λ and μ , and $\kappa = \kappa(s)$ is the curvature of $\partial\omega$. The first boundary layer profile satisfies $\varphi_s^1 = 0$ and

$$\varphi_{\natural}^1 = \kappa \zeta_n^0(s) \bar{\varphi}_{\natural}^m + \kappa \partial_n \zeta_3^0(s) \bar{\varphi}_{\natural}^b, \quad (7.2)$$

where $\bar{\varphi}_{\natural}^m = \bar{\varphi}_{\text{Mix1}, \natural}^m$ and $\bar{\varphi}_{\natural}^b = \bar{\varphi}_{\text{Mix1}, \natural}^b$ are typical profiles depending only on (t, x_3) .

The comparison between the above statement and Theorem 5.1 exhibit the strong influence of the curvature in the hard simply support situation: in any straight part of the boundary of ω , $\varphi^1 \equiv 0$.

This fact can be explained in a direct way: here all the tangential components of the displacement are set to zero, whereas the normal component of the traction is zero. Therefore, in the neighborhood $\mathcal{V}$ of any flat part of the lateral boundary Γ_0 , we can extend the displacement across Ω by reflection: the odd reflection of the tangential components and the even reflection of the normal component extend $\mathbf{u}(\varepsilon)$ to a new displacement $\tilde{\mathbf{u}}(\varepsilon)$ associated to extended horizontal boundary conditions $\tilde{\mathbf{g}}^\pm$ and volume forces $\tilde{\mathbf{f}}$. Thus the extended displacement satisfies estimates as if $\mathcal{V}$ is inside the plate, i.e. far away from the boundary layer.

If the support of the data avoids the lateral boundary Γ_0 in the neighborhood $\mathcal{V}$, then there are *no boundary layer terms at all* in $\mathcal{V}$: $\mathbf{u}(\varepsilon)$ can be expanded in a power series inside $\mathcal{V}$. The situation is exactly the same in the case of the ‘sliding edge condition’.

7.1 The traces of $\mathbf{u}_{\text{KL}}^0$

According to (3.18), $D_3^0 = 0$ yields $\zeta_3^0 = 0$ on $\partial\omega$, then $D_s^0 = 0$ is equivalent to $\zeta_s^0 = 0$ on $\partial\omega$. Next, $D_3^1 = 0$ yields $\zeta_3^1 = 0$ on $\partial\omega$, and $D_s^1 = 0$ provides the equation for φ_s^1

$$L_{\text{Dir}}(\varphi_s^1) = (0; 0; -\zeta_s^1). \quad (7.3)$$

Proposition 4.4 yields that the only exponentially decreasing solution is $\varphi_s^1 \equiv 0$ obtained with $\zeta_s^1 = 0$ on $\partial\omega$.

According to (3.24) T_n^0 and T_n^1 are zero. Conditions $T_n^2 = 0$, cf (3.25b), and $D_3^2 = 0$ yield that $\varphi_{\natural}^1$ has to solve

$$E_{\text{Mix1}}(\varphi_{\natural}^1) = (0; 0; -T_n^m(\zeta_*^0) + x_3 M_n(\zeta_3^0), -(\zeta_3^2 + v_3^2)). \quad (7.4)$$

With formulas (4.8) we can compute the three coefficients δ_3 , δ_5 and δ_7 , and determine conditions on $T_n^m(\zeta_*^0)$, $M_n(\zeta_3^0)$ and ζ_3^2 so that these three coefficients are zero, ensuring that $\varphi_{\natural}^1$ is exponentially decaying. We have

$$\bar{\gamma}_5 \delta_5 = \int_{-1}^{+1} -T_n^m(\zeta_*^0) + x_3 M_n(\zeta_3^0) dx_3 \quad (7.5a)$$

$$\bar{\gamma}_7 \delta_7 = \int_{-1}^{+1} x_3 T_n^m(\zeta_*^0) - x_3^2 M_n(\zeta_3^0) dx_3 \quad (7.5b)$$

$$\bar{\gamma}_3 \delta_3 = \int_{-1}^{+1} 6\bar{p}_3(-T_n^m(\zeta_*^0) + x_3 M_n(\zeta_3^0)) + 6\mu(\bar{p}_2 + \bar{p}_3')(\zeta_3^2 + v_3^2) dx_3. \quad (7.5c)$$

With (7.5a) and (7.5b), the conditions $\delta_5 = 0$ and $\delta_7 = 0$ give immediately that $T_n^m(\zeta_*^0) = 0$ and $M_n(\zeta_3^0) = 0$ on $\partial\omega$ respectively. Then with the formula $v_3^2 = \bar{p}_1 \operatorname{div}_* \zeta_*^0 + \bar{p}_2 \Delta_* \zeta_3^0$ we can compute from (7.5c)

$$\bar{\gamma}_3 \delta_3 = -4(\tilde{\lambda} + 2\mu) \left(\zeta_3^2 - \frac{\tilde{\lambda}}{30\mu} \Delta_* \zeta_3^0 \right),$$

whence the relation $30\mu \zeta_3^2 = \tilde{\lambda} \Delta_* \zeta_3^0$ ensuring the existence of a unique exponentially decreasing profile solution of (7.4).

But we have on $\partial\omega$

$$T_n^m(\zeta_*) = (\tilde{\lambda} + 2\mu) \operatorname{div}_* \zeta_* + 2\mu(\kappa \zeta_n - \partial_s \zeta_s) \quad (7.6a)$$

$$M_n(\zeta_3) = (\tilde{\lambda} + 2\mu) \Delta_* \zeta_3 + 2\mu(\kappa \partial_n \zeta_3 - \partial_{ss} \zeta_3). \quad (7.6b)$$

Since ζ_s^0 and ζ_3^0 are zero on $\partial\omega$, then $\partial_s \zeta_s^0$ and $\partial_{ss} \zeta_3^0$ are also zero and since $T_n^m(\zeta_*^0) = 0$ and $M_n(\zeta_3^0) = 0$ we deduce from (7.6) the relations

$$\operatorname{div}_* \zeta_*^0 = -\frac{2\mu}{\tilde{\lambda} + 2\mu} \kappa \zeta_n^0 \quad \text{and} \quad \Delta_* \zeta_3^0 = -\frac{2\mu}{\tilde{\lambda} + 2\mu} \kappa \partial_n \zeta_3^0. \quad (7.7)$$

Therefore, with $\bar{\varphi}_{\natural}^m$ the solution of

$$E_{\text{Mix1}}(\bar{\varphi}_{\natural}^m) = (0; 0; 0, \frac{2\mu}{\tilde{\lambda} + 2\mu} \bar{p}_1), \quad (7.8)$$

and with $\bar{\varphi}_{\natural}^b$ the solution of

$$E_{\text{Mix1}}(\bar{\varphi}_{\natural}^b) = (0; 0; 0, \frac{2\mu}{\tilde{\lambda} + 2\mu} (\frac{\tilde{\lambda}}{30\mu} + \bar{p}_2)), \quad (7.9)$$

we obtain the expression (7.2) of the first boundary layer term.

7.2 The traces of $\mathbf{u}_{\text{KL}}^1$

The next relations are deduced from $T_n^3 = 0$ and $D_3^3 = 0$ which yield that $\varphi_{\natural}^2$ has to solve

$$E_{\text{Mix1}}(\varphi_{\natural}^2) = \left(-(\mathcal{B}^{(1)}\varphi^1)_{\natural}; -(\mathcal{G}^{(1)}\varphi^1)_{\natural}; -T_t^{(1)}(\varphi^1) - T_n^{\text{m}}(\zeta_*^1) + x_3 M_n(\zeta_3^1), -(\zeta_3^3 + v_3^3) \right). \quad (7.10)$$

The cancellation of the coefficients δ_5 , δ_7 and δ_3 , cf (7.5) is ensured by relations determining $T_n^{\text{m}}(\zeta_*^1)$, $M_n(\zeta_3^1)$ and ζ_3^3 . In particular we have

$$\begin{aligned} T_n^{\text{m}}(\zeta_*^1) = & -\frac{1}{2} \left(\int_{\Sigma_+} (\mathcal{B}^{(1)}\varphi^1)_t(t, x_3) dt dx_3 \right. \\ & - \int_{\mathbb{R}_+} \left((\mathcal{G}^{(1)}\varphi^1)_t(t, 1) - (\mathcal{G}^{(1)}\varphi^1)_t(t, -1) \right) dt \\ & \left. + \int_{-1}^{+1} T_t^{(1)}(\varphi^1)(0, x_3) - x_3 M_n(\zeta_3^1)(0) dx_3 \right). \end{aligned} \quad (7.11)$$

Since $\varphi_s^1 = 0$, the terms in the right hand side of (7.10) reduce to

$$(\mathcal{B}^{(1)}\varphi^1)_t = -(\lambda + 2\mu)\kappa \partial_t \varphi_t^1, \quad (\mathcal{G}^{(1)}\varphi^1)_t = 0, \quad T_t^{(1)}(\varphi^1) = -\lambda \kappa \varphi_t^1$$

and

$$(\mathcal{B}^{(1)}\varphi^1)_3 = -\mu \kappa \partial_t \varphi_3^1 - (\lambda + \mu) \kappa \partial_3 \varphi_t^1, \quad (\mathcal{G}^{(1)}\varphi^1)_3 = -\lambda \kappa \varphi_t^1, \quad T_3^{(1)}(\varphi^1) = 0.$$

Whence the formulas (7.1) for $T_n^{\text{m}}(\zeta_*^1)$ and $M_n(\zeta_3^1)$ are deduced from the above relations and the expression (7.2) for $\varphi_{\natural}^1$.

This process of solving the asymptotics is clearly recursive.

8 SOFT SIMPLY SUPPORTED

The space of rigid motions $\mathcal{R}_{\textcircled{4}}$ is three-dimensional and spanned by the in-plane rigid motions.

Theorem 8.1 *The expansion (2.1)-(2.2) holds for the solution $\mathbf{u}(\varepsilon)$ of the scaled variant of problem (1.1) with lateral condition $\textcircled{4}$, compatibility condition (1.2) and orthogonality condition (1.3). The Kirchhoff-Love generators ζ_*^k and ζ_3^k are solutions of membrane and bending problems with interior data $\mathbf{R}_{\text{m}}^k$ and R_{b}^k , and boundary conditions prescribing $T_n^{\text{m}}(\zeta_*^k)$, $T_s^{\text{m}}(\zeta_*^k)$, and ζ_3^k , $M_n(\zeta_3^k)$ respectively, and with corresponding compatibility and orthogonality conditions. ζ^0 and ζ^1 satisfy the following boundary conditions on $\partial\omega$*

$$\begin{aligned} T_n^{\text{m}}(\zeta_*^0) &= 0, & T_s^{\text{m}}(\zeta_*^0) &= 0, & \zeta_3^0 &= 0, \\ & & & & M_n(\zeta_3^0) &= 0, \\ T_n^{\text{m}}(\zeta_*^1) &= c_1^{\textcircled{4}} \kappa \operatorname{div}_* \zeta_*^0, & T_s^{\text{m}}(\zeta_*^1) &= c_2^{\textcircled{4}} \partial_s \operatorname{div}_* \zeta_*^0, & \zeta_3^1 &= 0, \\ & & & & M_n(\zeta_3^1) &= (c_4^{\textcircled{4}} \kappa^2 + c_5^{\textcircled{4}} \partial_{ss}) \partial_n \zeta_3^0, \end{aligned} \quad (8.1)$$

where $c_1^{(4)}$ and $c_4^{(4)}$ are proportional to their homologues for the hard simple support: there holds $c_1^{(4)} = -\frac{1}{2\mu}(\tilde{\lambda} + 2\mu)c_1^{(3)}$ and $c_4^{(4)} = c_4^{(3)}$. Moreover $c_2^{(4)}$ is equal to $c_1^{(4)}$ and $c_5^{(4)}$ is another coefficient, only depending on λ and μ . The first boundary layer profile fulfills $\varphi_s^1 = \partial_{sn}\zeta_3^0(s)\bar{\varphi}_{\text{Neu}}^s$, where $\bar{\varphi}_{\text{Neu}}^s$ satisfies $L_{\text{Neu}}(\bar{\varphi}_{\text{Neu}}^s) = (0; 0; 2\mu x_3)$ and

$$\varphi_{\natural}^1 = -\frac{1}{2\mu}(\tilde{\lambda} + 2\mu) \operatorname{div}_* \zeta_*^0(s) \bar{\varphi}_{\natural}^m + \kappa \partial_n \zeta_3^0(s) \bar{\varphi}_{\natural}^b, \quad (8.2)$$

where $\bar{\varphi}_{\natural}^m = \bar{\varphi}_{\text{Mix1},\natural}^m$ and $\bar{\varphi}_{\natural}^b = \bar{\varphi}_{\text{Mix1},\natural}^b$ are the profiles introduced in (7.2), compare (7.8) and (7.9).

8.1 The traces of u_{KL}^0

According to (3.18), $D_3^0 = 0$ and $D_3^1 = 0$ yields $\zeta_3^0 = 0$ and $\zeta_3^1 = 0$ on $\partial\omega$. According to (3.24) T_s^0 and T_s^1 are zero. Condition $T_s^2 = 0$, yields cf (3.25b)

$$T_s^{(0)}(\varphi^1) = -T_s^m(\zeta_*^0) + 2\mu x_3 (\partial_n + \kappa) \partial_s \zeta_3^0.$$

Taking into account the already known trace condition $\zeta_3^0 = 0$, we obtain that φ_s^1 solves the Laplace Neumann problem on the half-strip:

$$L_{\text{Neu}}(\varphi_s^1) = (0; 0; -T_s^m(\zeta_*^0) + 2\mu x_3 \partial_{sn} \zeta_3^0). \quad (8.3)$$

Proposition 4.5 yields that the only exponentially decreasing solution φ_s^1 is obtained with $T_s^m(\zeta_*^0) = 0$ on $\partial\omega$. Moreover, φ_s^1 is equal to $\partial_{sn}\zeta_3^0(s)\bar{\varphi}_{\text{Neu}}^s$, where $\bar{\varphi}_{\text{Neu}}^s$ is the unique exponentially decaying solution of $L_{\text{Neu}}(\bar{\varphi}_s) = (0; 0; 2\mu x_3)$ and therefore is clearly odd with respect to the variable x_3 .

According to (3.24) T_n^0 and T_n^1 are zero. Just like in the hard simple support case, conditions $T_n^2 = 0$ and $D_3^2 = 0$ yield that $\varphi_{\natural}^1$ has to solve equation (7.4) and we obtain the conditions $T_n^m(\zeta_*^0) = 0$ and $M_n(\zeta_3^0) = 0$ on $\partial\omega$.

Since ζ_3^0 is zero on $\partial\omega$, then $\partial_{ss}\zeta_3^0$ is also zero and we deduce from $M_n(\zeta_3^0) = 0$ the relation

$$\Delta_* \zeta_3^0 = -\frac{2\mu}{\tilde{\lambda} + 2\mu} \kappa \partial_n \zeta_3^0. \quad (8.4)$$

Therefore, similarly as in §7.1 we obtain the expression (8.2) of the first boundary layer term.

Since the rigid motions belonging to $\mathcal{R}_{\textcircled{4}}$ do not depend on the vertical variable x_3 , the compatibility condition ensuring the existence of ζ_*^0 is a consequence of the expression of the right hand side $\mathbf{R}_m^0$ and of the compatibility conditions (1.2) assumed for the data $\mathbf{f}$ and $\mathbf{g}^{\pm}$ of the three-dimensional problem.

8.2 The traces of u_{KL}^1

Concerning the tangential component, the condition $T_s^3 = 0$ yields in the same manner as in §6.2 that φ_s^2 has to solve

$$L_{\text{Neu}}(\varphi_s^2) = \left(-(\mathcal{B}^{(1)}\varphi^1)_s; -(\mathcal{G}^{(1)}\varphi^1)_s; -T_s^m(\zeta_*^1) + 2\mu x_3 \partial_{sn} \zeta_3^1 - T_s^{(1)}(\varphi^1) \right). \quad (8.5)$$

Proposition 4.5 yields that φ_s^2 is exponentially decreasing if and only if (6.4) holds. In this case, the terms in (6.4) are given by

$$\begin{aligned} (\mathcal{B}^{(1)}\varphi^1)_s &= \mu \kappa (\partial_{tt} + \partial_{33})(t\varphi_s^1) - \mu \kappa \partial_t \varphi_s^1 + (\lambda + \mu) \partial_s (\partial_t \varphi_t^1 + \partial_3 \varphi_3^1), \\ (\mathcal{G}^{(1)}\varphi^1)_s &= \mu \partial_s \varphi_3^1, \quad T_s^{(1)}(\varphi^1) = \mu (\partial_s \varphi_t^1 + 2\kappa \varphi_s^1). \end{aligned}$$

Taking into account the parities of the first boundary layer terms, we see as in §6 that only the boundary layer term $\operatorname{div}_* \zeta_*^0(s) \bar{\varphi}_{\natural}^m(t, x_3)$ must be taken into consideration. The same calculations as in §6.2 then lead to the expression for $T_s^m(\zeta_*^1)$.

The next relations are deduced from $T_n^3 = 0$ and $D_3^3 = 0$ which lead to the following problem for $\varphi_{\natural}^2$

$$E_{\text{Mix1}}(\varphi_{\natural}^2) = \left(-(\mathcal{B}^{(1)}\varphi^1)_{\natural}; -(\mathcal{G}^{(1)}\varphi^1)_{\natural}; -T_t^{(1)}(\varphi^1) - T_n^m(\zeta_*^1) + x_3 M_n(\zeta_3^1), -(\zeta_3^3 + v_3^3) \right). \quad (8.6)$$

The cancellation of the coefficients δ_5 , δ_7 and δ_3 , cf (7.5), ensuring the existence of exponentially decreasing profiles is guaranteed by relations determining $T_n^m(\zeta_*^1)$, $M_n(\zeta_3^1)$ and ζ_3^3 . The terms in the right hand side of (8.6) are given by

$$\begin{aligned} (\mathcal{B}^{(1)}\varphi^1)_t &= -(\lambda + 2\mu) \kappa \partial_t \varphi_t^1 + (\lambda + \mu) \partial_t \partial_s \varphi_s^1, \\ (\mathcal{G}^{(1)}\varphi^1)_t &= 0, \quad T_t^{(1)}(\varphi^1) = \lambda (\partial_s \varphi_s^1 - \kappa \varphi_t^1) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{B}^{(1)}\varphi^1)_3 &= -\mu \kappa \partial_t \varphi_3^1 + (\lambda + \mu) \partial_3 (-\kappa \varphi_t^1 + \partial_s \varphi_s^1), \\ (\mathcal{G}^{(1)}\varphi^1)_3 &= \lambda (-\kappa \varphi_t^1 + \partial_s \varphi_s^1), \quad T_3^{(1)}(\varphi^1) = 0. \end{aligned}$$

Taking into account the parities of the boundary layer terms, the formulas in (8.1) for $T_n^m(\zeta_*^1)$ ($\delta_5 = 0$) and $M_n(\zeta_3^1)$ ($\delta_7 = 0$) are deduced in a similar way as in §7.2. Moreover, a relation determining ζ_3^3 can be derived at this stage (compare the expression for the constant δ_3 in (7.5c)). By evaluating in the condition $\delta_7 = 0$ the terms involving $\partial_{ssn} \zeta_3^0$, an expression for the constant $c_5^{(4)}$ can be given. The derivation of this expression reads

$$\begin{aligned} c_5^{(4)} &= \frac{3}{2} \left[(\lambda + \mu) \int_{\Sigma^+} (x_3 \partial_t \bar{\varphi}_{\text{Neu}}^s - t \partial_3 \bar{\varphi}_{\text{Neu}}^s) dt dx_3 + \lambda \int_0^\infty t \bar{\varphi}_{\text{Neu}}^s(1, t) dt \right. \\ &\quad \left. - \lambda \int_0^\infty t \bar{\varphi}_{\text{Neu}}^s(-1, t) dt + \lambda \int_{-1}^{+1} x_3 \bar{\varphi}_{\text{Neu}}^s(0, x_3) dx_3 \right] \\ &= \frac{3}{2} \left[-\mu \int_{-1}^{+1} x_3 \bar{\varphi}_{\text{Neu}}^s(0, x_3) dx_3 - 2\mu \int_0^\infty t \bar{\varphi}_{\text{Neu}}^s(1, t) dt \right]. \end{aligned}$$

Applying the second Green's formula for Laplace on rectangles $\Sigma_L = (0, L) \times (-1, 1)$ to the functions $\bar{\varphi}_{\text{Neu}}^s(t, x_3)$ and $w(t, x_3) = t x_3$ and considering $L \rightarrow +\infty$, the relation

$$\int_{-1}^{+1} x_3 \bar{\varphi}_{\text{Neu}}^s(0, x_3) dx_3 = 2 \int_0^\infty t \bar{\varphi}_{\text{Neu}}^s(t, 1) dt$$

is obtained, and thus

$$c_5^{(4)} = -3\mu \int_{-1}^{+1} x_3 \bar{\varphi}_{\text{Neu}}^s(0, x_3) dx_3$$

is valid.

Concerning the generator ζ^1 of $\mathbf{u}_{\text{KL}}^1$, the last thing we have to check is the compatibility condition for the data. This concerns only ζ_*^1 and the non-zero data are only the traces of the traction $T_n^m(\zeta_*^1) = c_1^{(4)} \kappa \operatorname{div}_* \zeta_*^0$ and $T_s^m(\zeta_*^1) = c_2^{(4)} \partial_s \operatorname{div}_* \zeta_*^0$. We have to check that for any in-plane rigid motion $\mathbf{v}_*$ (whose normal and tangential components are denoted by v_n and v_s as usual)

$$\int_{\partial\omega} \left(T_n^m(\zeta_*^1) v_n + T_s^m(\zeta_*^1) v_s \right) ds = 0. \quad (8.7)$$

We are going to give two proofs of (8.7) by two different methods. The first one deals with the explicit expressions of $T_n^m(\zeta_*^1)$ and $T_s^m(\zeta_*^1)$ and derives the identity $c_1^{(4)} = c_2^{(4)}$ by direct computations. The second one goes back to the whole construction. Its advantage is that it can be extended by recursivity to the compatibility conditions necessary to the construction of the next terms ζ_*^k , $k = 2, \dots$ in the asymptotics.

FIRST PROOF. With the help of an integration by parts and Frenet's formulas, it is straightforward to obtain

Lemma 8.2 *Let g belong to $H^{1/2}(\partial\omega)$. Then for any rigid motion $\mathbf{v}_* = a(-x_2, x_1) + (b, c)$, we have*

$$\int_{\partial\omega} (\kappa g v_n + \partial_s g v_s) ds = 0. \quad (8.8)$$

Thus it remains to check that $c_1^{(4)} = c_2^{(4)}$. Applying Proposition 4.8 to equation (8.6), we obtain from the cancellation of the coefficient δ_5 that

$$c_1^{(4)} = -\frac{\tilde{\lambda} + 2\mu}{4\mu} \left[(\lambda + 2\mu) \int_{\Sigma^+} \partial_t \bar{\varphi}_t^m dt dx_3 + \lambda \int_{-1}^{+1} \bar{\varphi}_t^m(0, x_3) dx_3 \right]. \quad (8.9)$$

Applying Proposition 4.5 to equation (8.5), we obtain that

$$c_2^{(4)} = -\frac{\tilde{\lambda} + 2\mu}{4\mu} \left[-(\lambda + \mu) \int_{\Sigma^+} (\partial_t \bar{\varphi}_t^m + \partial_3 \bar{\varphi}_3^m) dt dx_3 + 2\mu \int_0^\infty \bar{\varphi}_3^m(t, 1) dt - \mu \int_{-1}^{+1} \bar{\varphi}_t^m(0, x_3) dx_3 \right]. \quad (8.10)$$

Integrating by parts, we transform (8.9) and (8.10) into

$$c_1^{(4)} = -\frac{\tilde{\lambda} + 2\mu}{4\mu} \left[2\mu \int_{\Sigma^+} \partial_t \bar{\varphi}_t^m dt dx_3 \right] \quad (8.11)$$

and

$$c_2^{(4)} = -\frac{\tilde{\lambda} + 2\mu}{4\mu} \left[-\lambda \int_{\Sigma^+} (\partial_t \bar{\varphi}_t^m + \partial_3 \bar{\varphi}_3^m) dt dx_3 \right]. \quad (8.12)$$

But since $\bar{\varphi}_1^m$ solves $E_{\text{Mix1}}(\bar{\varphi}_1^m) = (0; 0; 0, c p_1)$, integrating versus t on Σ^+ the relation $\mu(\partial_{tt} + \partial_{33})\bar{\varphi}_t^m + (\lambda + \mu)\partial_t(\partial_t \bar{\varphi}_t^m + \partial_3 \bar{\varphi}_3^m) = 0$ and using the boundary condition $\partial_3 \bar{\varphi}_t^m + \partial_t \bar{\varphi}_3^m = 0$ on the sides $x_3 = \pm 1$, we find that

$$\int_{\Sigma^+} ((\lambda + 2\mu)\partial_t \bar{\varphi}_t^m + \lambda \partial_3 \bar{\varphi}_3^m) dt dx_3 = 0. \quad (8.13)$$

Whence the equality $c_1^{(4)} = c_2^{(4)}$.

SECOND PROOF. We set $\varphi = \varphi^1 + \varepsilon \varphi^2$. Denoting by $\mathbf{T}_*^m$ the membrane traction, we have by construction

$$\mathbf{T}_*^m(\zeta_*^0 + \varepsilon \zeta_*^1) = -\frac{1}{2} \left(\int_{\Sigma^+} \mathbf{f}_*(\varepsilon) - \int_{\mathbb{R}^+} (\mathbf{g}_*^+(\varepsilon) - \mathbf{g}_*^-(\varepsilon)) + \int_{-1}^{+1} \mathbf{h}_*(\varepsilon) \right), \quad (8.14)$$

compare with (6.4), (7.11), where

$$\mathbf{f}(\varepsilon) = \mathcal{B}\varphi + \mathcal{O}(\varepsilon^2), \quad \mathbf{g}(\varepsilon) = \mathcal{G}\varphi + \mathcal{O}(\varepsilon^2), \quad \mathbf{h}(\varepsilon) = \mathbf{T}\varphi + \mathcal{O}(\varepsilon^2).$$

Setting on Ω^ε , $\mathbf{w}(\tilde{x}) = \chi(r) \varphi(\frac{r}{\varepsilon}, s, \frac{\tilde{x}_3}{\varepsilon})$ and integrating (8.14) along $\partial\omega$ we obtain for any rigid motion $\mathbf{v}$ in $\mathcal{R}_{(4)}$

$$\int_{\partial\omega} \mathbf{T}_*^m(\zeta_*^0 + \varepsilon \zeta_*^1) \cdot \mathbf{v}_* = \frac{1}{2} \int_{\Omega^\varepsilon} A e(\mathbf{w}) : e(\mathbf{v}) + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),$$

whence (8.7).

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