

WEYL'S FORMULA FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS WITH NEGATIVE ORDER ON $L^2(\mathbb{R}^n)$

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Introduction

The starting point of this work is a paper by Birman and Solomjak [BI-SO 1] in which they study the eigenvalues asymptotics of a class of integral compact operators on $L^2(\mathbb{R}^n)$. These operators are pseudodifferential, with symbols that are homogeneous (or quasi-homogeneous) of negative order with respect to the phase variable ; there are very few regularity assumptions with respect to the space variable.

If such an operator A is self-adjoint, $(\lambda_j^+(A))$ denoting the positive eigenvalues decreasing sequence of A , [BI-SO 1] gives an equivalent to the $\lambda_j^+(A)$, by powers of j when $j \rightarrow +\infty$, and the corresponding result for negative eigenvalues (cf § 2.A). The method of [BI-SO 1] is based on Courant's mini-max principle and consists in reducing to a model problem on the torus $\mathbb{T}^n$.

In this work, we intend to extend and precise [BI-SO 1] results in two directions :

- (1) relaxing the homogeneity assumption
- (2) getting a remainder estimate.

As the symbol may vanish and change of sign, there are some difficulties we will overcome by making an hypoelliptic regularization of the operators and then using a Mellin fonctionnal pseudodifferential calculus.

Here is the structure of our paper

§1 : Assumptions and the main result.

§2 : Special cases and examples.

§3 : Further information about spectral theory of globally hypoelliptic pseudodifferential operators on $\mathbb{R}^n$.

§4 : Proof of the main result.

Annex A : Weyl-Ky Fan inequalities.

Annex B : Composition formula with precised remainder.

Annex C : Computation for an example.

The main results of this paper have been announced in the authors' note [DA-RO].

§1 : Assumptions and the main result

Our operators are described by the properties of their Weyl's symbols, according to L. Hörmander's formalization [HO 1]. For the sake of simplicity, we restrict ourselves to the diagonal metrics. Let us recall some definitions.

On $T^* \mathbf{R}^n$, identified to $\mathbf{R}^{2n} = \mathbf{R}_x^n \times \mathbf{R}_\xi^n$, let ϕ and φ be two weight functions, with their values in $]0, +\infty[$.

(1.1) Definition

Let m be a weight function, $m : \mathbf{R}^{2n} \rightarrow]0, +\infty[$.

- (1) If there exists $C_0, C_1 > 0$ such that :

$$|y| \phi(x, \xi) + |\eta| \varphi(x, \xi) \leq C_0 (\phi \varphi)(x, \xi) \Rightarrow C_1^{-1} m(x, \xi) \leq m(x+y, \xi+\eta) \leq C_1 m(x, \xi)$$

then, m is called (ϕ, φ) -continuous.

- (2) If, moreover, there exists $C, M > 0$ such that :

$$m(x+y, \xi+\eta) \leq C m(x, \xi) [1 + |y| \phi(x, \xi) + |\eta| \varphi(x, \xi)]^M$$

then, m is called (ϕ, φ) -temperate.

In this whole work, we assume :

- (H₁) ϕ^{-1}, φ^{-1} are bounded on $\mathbf{R}^{2n}$ and they are (ϕ, φ) -continuous and $(1, 1)$ -temperate.

If m is a (ϕ, φ) -temperate weight, according to [BE] we denote by $S(m ; \phi, \varphi)$ the class of symbols $a \in C^\infty(\mathbf{R}^{2n})$ such that for all $k \in \mathbf{N}$ the following quantities are finite :

$$(1.2) \quad s_k(a ; m) = \sup_{\mathbf{R}^n, |\alpha+\beta|=k} |m^{-1} \phi^{|\alpha|} \varphi^{|\beta|} \partial_\xi^\alpha \partial_x^\beta a|.$$

The (s_k) are a family of semi-norms for $S(m ; \phi, \varphi)$.

To a symbol a in $S(m ; \phi, \varphi)$, we associate the operator $A = \text{Op}^W a$, according to the Weyl's quantization :

$$Au(x) = \int e^{i\langle x-y, \xi \rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi$$

with $d\xi = (2\pi)^{-n} d\xi$.

Here are complementary assumptions about the weights :

$$(H_2) \quad \exists C, \varepsilon_0 > 0 \text{ such that } C(1+|x|+|\xi|)^{\varepsilon_0} \leq (\phi \varphi)(x, \xi) \text{ on } \mathbf{R}^{2n}$$

$$(W) \quad m \text{ is } (\phi, \varphi)\text{-temperate and } m \in S(m ; \phi, \varphi)$$

$$(N) \quad \exists K, K' > 0, \exists \gamma, \gamma' > 0 \text{ such that } K' m^{\gamma'} \leq (\phi \varphi)^{-1} \leq K m^\gamma \text{ on } \mathbf{R}^{2n}.$$

With (H_2) , (N) states that m has a negative order.

Let be $a \in S(m ; \phi, \varphi)$ with real values. (H_1) , (H_2) , (W) and (N) imply that $\text{Op}^W a$ is self-adjoint and compact, with the kernel :

$$K_a(x, y) = \int e^{i\langle x-y, \xi \rangle} a\left(\frac{x+y}{2}, \xi\right) d\xi.$$

The spectrum of A is discrete with only one accumulation point in 0 . Denote by $\lambda_j^+(A)$ (resp. $\lambda_j^-(A)$) the decreasing (resp. increasing) sequence of positive (resp. negative) eigenvalues of A , repeated according to their multiplicities. For $\lambda > 0$, we introduce the counting functions :

$$N_{\pm}(\lambda; A) = \# \{j \in \mathbf{N} / \pm \lambda_j^{\pm}(A) > \lambda\}.$$

Like the classical Weyl's formula for the Dirichlet's problem on a bounded domain, we compare $N_{\pm}(\lambda; A)$ when $\lambda \rightarrow +\infty$, with the volume functions $V_{\pm}(\lambda; a)$ in the classical phase space :

$$V_{\pm}(\lambda; a) = \int_{\pm a > \lambda} dx d\xi.$$

For the functions V , we need, as in [TU-SU], a control of the derivative : for a decreasing function $f :]0, +\infty[\rightarrow]0, +\infty[$, we say that f verifies (T) if there exists $\lambda_0 > 0$, $\gamma_1, \gamma_2 > 0$ such that :

$$(T) \begin{cases} f \text{ is derivable on }]0, \lambda_0] \\ \gamma_1 f(\lambda) < -\lambda f'(\lambda) < \gamma_2 f(\lambda) \text{ for all } \lambda \in]0, \lambda_0]. \end{cases}$$

By integrating (T), we obtain the useful inequality :

$$(T') \quad \left(\frac{\lambda_2}{\lambda_1}\right)^{\gamma_1} < \frac{f(\lambda_1)}{f(\lambda_2)} < \left(\frac{\lambda_2}{\lambda_1}\right)^{\gamma_2} \quad \text{for } 0 < \lambda_1 < \lambda_2 < \lambda_0$$

Our main result is :

(1.3) Theorem

With the hypotheses (H_1) , (H_2) , (W) and (N) , let $a \in S(m; \phi, \varphi)$ be a real symbol and $A = \text{Op}^W a$. We assume that the volume functions $V_+(\cdot; m)$, $V_+(\cdot; a)$ and $V_-(\cdot; a)$ verify (T). Then, there is $\zeta > 0$ such that :

$$(1) \quad N_+(\lambda; A) = V_+(\lambda; a) + O(\lambda^{\zeta} V_+(\lambda; m)) \text{ when } \lambda \rightarrow 0_+;$$

$$(2) \text{ if moreover, } V_+(\lambda; m) = O(V_+(\lambda; a)), \text{ we have :}$$

$$(1.4) \quad N_+(\lambda; A) = V_+(\lambda; a) (1 + O(\lambda^{\zeta})) \text{ when } \lambda \rightarrow 0_+;$$

$$(1.5) \quad \lambda_j^+(A) = \Lambda^+(j; a) (1 + O(j^{-\zeta})) \text{ when } j \rightarrow +\infty$$

where $\Lambda^+(t; a)$ is the inverse function of $V_+(\cdot; a)$, which is well defined for t large enough.

We have corresponding statements linking N_- and V_- .

(1.6) Remarks

(1) The condition (T) about $V_-(\cdot; a)$ may be replaced by :

$$V_-(\lambda; a) = O(\lambda^{\zeta_0} V_-(\lambda; m)) \text{ with } \zeta_0 > 0.$$

(2) If $a = a_0 + a_1$ with $a_1 \in S(m(\phi, \varphi)^{-\epsilon}; \phi, \varphi)$ and $\epsilon > 0$, $V_{\pm}(\lambda; a)$ may be replaced by $V_{\pm}(\lambda; a_0)$

Asymptotics of (1.4) type are well known for wide classes of globally hypoelliptic pseudodifferential operators with positive order - cf §3 - Among the numerous works about this, let us quote : [TU-SU], [RO], [HO 2], [HE-RO 2], [MO], [FE]. However, in our case, the symbol need not to have a constant sign and we cannot reduce our operators to those of the above works : in particular, the separation between positive and negative parts of the spectrum is awkward.

About a connected problem, G. Grubb [GR] has noted that an hypoelliptic regularization can yield a result for positive operators. Our method of proof consists in sharpening this remark. We shall be led to use accurate results about functional calculus for a class of hypoelliptic operators (3.12) and we shall deduce from that a Weyl's formula of [HO 2] type (3.21). Those results will be stated in §3 and, moreover, we will show that they have other applications (3.16), (3.17), (3.20).

§2 Special cases and examples

2.A. Homogeneous symbols

1st case : $a = a_0 + a_1$ where a_0 is homogeneous of degree $\sigma < 0$ with respect to ξ variable for $|\xi| > 1/2$, and we suppose that there is some $s < \sigma$ such that :

- (i) $\partial_x^\alpha a_0(x, \xi) \leq C_\alpha (1+|x|)^{s-|\alpha|} \quad \forall \alpha \in \mathbb{N}^n, x \in \mathbb{R}^n, |\xi| = 1$
- (ii) $\partial_x^\alpha \partial_\xi^\beta a_1(x, \xi) \leq C_{\alpha, \beta} (1+|x|)^{s-|\alpha|} (1+|\xi|)^{\beta-|\beta|} \quad \forall \alpha, \beta \in \mathbb{N}^n, (x, \xi) \in \mathbb{R}^{2n}$

We can apply our result with the weights :

$$\phi(x, \xi) = 1+|\xi|; \quad \varphi(x, \xi) = 1+|x|; \quad m(x, \xi) = (1+|x|^2)^{s/2} (1+|\xi|^2)^{\sigma/2}.$$

For $A = \text{Op}^W a$, we obtain :

$$(2.1) \quad N_\pm(\lambda; A) = \lambda^{n/\sigma} (\gamma_\pm + O(\lambda^\zeta)), \quad \lambda \rightarrow 0, \text{ with } \zeta > 0$$

where :

$$(2.2) \quad \gamma_\pm = \frac{(2\pi)^{-n}}{n} \iint_{\mathbb{R}^n \times S^{n-1}} a_0^\pm(x, \xi)^{-n/\sigma} dx d\xi.$$

Note that the condition $s < \sigma$ yields the integrability of $a_0^\pm(x, \xi)^{-n/\sigma}$ on $\mathbb{R}^n \times S^{n-1}$.

(2.1) is equivalent to :

$$\lambda_j^\pm(A) = j^{\sigma/n} (C_\pm + O(j^{\zeta \sigma/n})), \quad j \rightarrow +\infty, \text{ where } C_\pm = (\gamma_\pm)^{-\sigma/n}.$$

We can compare those results with [BI-SO 1] ones :

there are the following differences :

- [BI-SO 1] symbols satisfy low regularity assumptions
- they only get an equivalent
- they require supplementary conditions about the behavior with respect to x variable :

$$\begin{aligned} \bullet s = -\infty & \quad \text{if } \sigma \leq -n \\ \bullet s < -n/2 & \quad \text{if } \sigma > -n/2. \end{aligned}$$

2^d case : we have obviously the same type of result if we permutate x and ξ roles. We are now interested with the case when the homogeneities with respect to x and ξ are equal to each other :

$$a(x, \xi) = b(x) c(\xi)$$

where b and c are homogeneous of degree $\sigma < 0$ outside the ball with radius $1/2$. We will compute in annex C the volume functions of a . Thus, for $\zeta > 0$.

$$(2.1') \quad N_{\pm}(\lambda ; A) = \lambda^{n/\sigma} \text{Log} \frac{1}{\lambda} \gamma_{\pm} + \lambda^{n/\sigma} \delta_{\pm} + O(\lambda^{n/(\sigma+\zeta)}), \lambda \rightarrow 0_+$$

where :

$$(2.2') \quad \gamma_{\pm} = \frac{(2\pi)^{-n}}{n!|\sigma|} \int_{S^{n-1}} b^{\pm}(\tilde{x})^{-n/\sigma} d\tilde{x} \cdot \int_{S^{n-1}} c^{\pm}(\tilde{\xi})^{-n/\sigma} d\tilde{\xi}$$

and $\delta_{\pm}$ may be computed as a function of b and c (annex C).

Note that a is not an homogeneous function of degree 2σ outside a neighborhood of 0 in $\mathbb{R}^{2n}$. Let us compare with the following case :

3rd case : let $\omega = (x, \xi)$. We suppose that a is C^{∞} and such that, for $|\omega| \geq 1/2$:

$$a(\omega) = |\omega|^{2\sigma} a\left(\frac{\omega}{|\omega|}\right). \text{ Then :}$$

$$(2.1'') \quad N_{\pm}(\lambda ; A) = \lambda^{n/\sigma} (\gamma_{\pm} + O(\lambda^{\zeta})) \quad \text{with } \epsilon > 0$$

$$(2.2'') \quad \gamma_{\pm} = \frac{(2\pi)^{-n}}{2n} \int_{S^{2n-1}} a^{\pm}(\tilde{\omega})^{-n/\sigma} d\tilde{\omega}.$$

Note that the 2^d and 3rd cases are not in the scope of [BI-SO 1].

2.B : Condition (T) about the volume functions

The condition (T) allows us to get rid of the homogeneity hypothesis. Here are assumptions about a under which condition (T) about $V_{\pm}(\lambda ; a)$ is fulfilled.

- (i) $a \in C^1(\mathbb{R}^{2n})$ and $a(x, \xi) \rightarrow 0$ when $|x| + |\xi| \rightarrow +\infty$
- (ii) there is a vector field X on $\mathbb{R}^{2n}$ such that :
 - (ii₁) $\exists \eta_1, \eta_2 > 0 : \eta_1 a \leq \langle X, \nabla a \rangle \leq \eta_2 a$
 - (ii₂) $\exists \delta_1, \delta_2 > 0 : \delta_1 \leq -\text{div } X \leq \delta_2$.

According to a classical result about the Leray's form on an hypersurface, we have, with dS_{λ} denoting the euclidian measure on $\{a = \lambda\}$:

$$-\lambda V'_{+}(a ; \lambda) = \int_{a=\lambda} \lambda \frac{dS_{\lambda}}{|\nabla a|} = \int_{a=\lambda} a \frac{dS_{\lambda}}{|\nabla a|}.$$

With (ii₁) that gives :

$$(2.3) \quad \frac{1}{\eta_2} \int_{a=\lambda} \langle X, \nabla a \rangle \frac{dS_{\lambda}}{|\nabla a|} \leq -\lambda V'_{+}(a ; \lambda) \leq \frac{1}{\eta_1} \int_{a=\lambda} \langle X, \nabla a \rangle \frac{dS_{\lambda}}{|\nabla a|}.$$

With the Green's formula, we have :

$$\int_{a=\lambda} \langle X, \nabla a \rangle \frac{d S_\lambda}{|\nabla a|} = - \int_{a \geq \lambda} \operatorname{div} X \, dx \, d\xi$$

with (ii₂) and (2.3), that yields condition (T) about $V_\pm(\lambda; a)$.

Note that if a is quasi-homogeneous, with weights k_i on x_i and h_i on ξ_i , we may take :

$$F(x, \xi) = -(k_1 x_1, \dots, k_n x_n; h_1 \xi_1, \dots, h_n \xi_n)$$

2.C : Schrödinger operators

Let us consider : $A = (-\Delta + E)^{\sigma/2} V(-\Delta + E)^{\sigma/2}$ with Δ the Laplace's operator on $\mathbf{R}^n$, $E > 0$, $\sigma < 0$, and the potential V such that there is $s < 0$:

$$(2.4) \quad |\partial^\alpha V(x)| \leq C_\alpha (1+|x|)^{2s-|\alpha|}$$

If $s < \sigma$, this example is a particular case of 2.A,1. Thus, let us assume that $\sigma < s$.

According to remark (1.6/2) we may consider the " principal symbol " a_0 of A instead of the complete symbol.

$$a_0(x, \xi) = (|\xi|^2 + E)^\sigma V(x)$$

According to 2.B, the condition (T) will be fulfilled for $V_\pm(\lambda; a_0)$ if, for $C, C', R > 0$, $k_1, \dots, k_n > 0$, we have :

$$(2.5) \quad C V(x) \leq - \sum_{i=1}^n k_i x_i \partial_{x_i} V(x) \leq C' V(x), \text{ pour } |x| > R$$

In that case, we have :

$$N_\pm(\lambda; A) = V_\pm(\lambda; a_0) + O(\lambda^\zeta f_{s, \sigma}(\lambda))$$

where $f_{s, \sigma}(\lambda) = \lambda^{n/2s}$ if $s < \sigma$ and $f_{s, \sigma}(\lambda) = \lambda^{n/2s} \operatorname{Log} \frac{1}{\lambda}$ if $s = \sigma$.

As an application, let us consider the stationary problem for Schrödinger equation :

$$(2.6) \quad (\Delta + gV) \Psi = E \Psi$$

where $E > 0$ is the energy and $g \in \mathbf{R}$ is a coupling constant. For a fixed E , we search values of g for which (2.6) has a non-null solution Ψ in $D(\Delta) \cap D(V)$. For V verifying (2.4) and $g \neq 0$, (2.6) is equivalent to

$$(2.7) \quad \Psi \in L^2(\mathbf{R}^n) \setminus \{0\}, (-\Delta + E)^{-1/2} V(-\Delta + E)^{-1/2} \Psi = \frac{1}{g} \Psi$$

It is the above case, with $\sigma = -1$. Thus, (2.6) has non-trivial solutions for two sequences $(g_k^+(E))$ and $(g_k^-(E))$ where $(g_k^+(E))$ is positive, increasing and $(g_k^-(E))$ is negative, decreasing. If infinite, each of those sequences are not bounded. If V verifies (2.5), with :

$$N_\pm(g, E) = \# \{k/g_k^\pm(E) \geq \pm g\} \quad (g > 0)$$

we have, when $g \rightarrow +\infty$:

$$(2.8) \quad N_{\pm}(g, E) = \frac{(2\pi)^{-n}}{n} \Gamma_n \int (g V(x) - E)_{\pm}^{n/2} dx + 0(g^{-\zeta} \int (g(1+|x|^2)^{\zeta} - E)^{n/2} dx$$

where Γ_n is the volume of the unit ball of $\mathbf{R}^n$.

Many works have given formulas of (2.8) type ([SI], [MA]).

2.D : Equation $A\Psi = \lambda B\Psi$.

Let $A = \text{Op}^W a$ and $B = \text{Op}^W b$ be two self-adjoint operators on $L^2(\mathbf{R}^n)$, possibly unbounded with domains $D(A)$ and $D(B)$. With (ϕ, φ) weight functions verifying (H_1) , (H_2) , we suppose that :

- (i) $\left\{ \begin{array}{l} A \text{ is positive and invertible,} \\ a = a_0 + a_1 \text{ where } a_0 \text{ is a temperate weight such that} \\ a_0 \in S(a_0; \phi, \varphi) \text{ and } a_1 \in S(a_0(\phi\varphi)^{-\varepsilon}; \phi, \varphi) \text{ with } \varepsilon > 0 \end{array} \right.$
- (ii) $\left\{ \begin{array}{l} b \in S(q; \phi, \varphi) \text{ where } q \text{ is a temperate weight such that} \\ q \in S(q; \phi, \varphi); \\ m := a_0^{-1} q \text{ verifies the hypotheses (W) and (N).} \end{array} \right.$

And we consider the spectral problem :

$$(2.9) \quad A\Psi = \lambda B\Psi, \quad \Psi \in D(A) \cap D(B) \setminus \{0\}.$$

By mean of composition by the operator $C = A^{-1/2}$, (2.9) is equivalent to :

$$\Psi = \lambda C B C \Psi, \quad \Psi \in L^2(\mathbf{R}^n) \setminus \{0\}$$

Thanks to (ii), theorem (1.3) can be applied to $C B C$, and if the volume functions satisfy the condition (T), we obtain, for a $\zeta > 0$, when $\lambda \rightarrow +\infty$:

$$(2.10) \quad N_{\pm}(\lambda; A, B) = (2\pi)^{-n} \iint_{a(x, \xi) < \pm \lambda b(x, \xi)} dx d\xi + 0(\lambda^{-\zeta} \iint_{a(x, \xi) < \lambda q(x, \xi)} dx d\xi)$$

where $N_{+}(\lambda; A, B)$ (resp. $N_{-}(\lambda; A, B)$) is the number of eigenvalues of (2.9) belonging to $]0, \lambda]$ (resp. $]-\lambda, 0[$).

Many papers are devoted to Weyl's formulas of (2.10) type, with various hypotheses (see [BI-SO 3] and [FL-LA], as well as their bibliographies). Most of them suppose that A is an elliptic differential operator and B is the multiplication by a function ρ . In particular, for Fleckinger and Lapidus in [FL-LA], A is of Schrödinger's type and ρ may be discontinuous (for instance, the characteristic function of a compact set). However, those works generally only give an equivalent. Let us note that we find the same result as the one announced by Boitmatov and Kostyuchenko in [BO-KO] under assumptions that seem to us more general and more natural.

§3 Further information about spectral theory of globally hypoelliptic pseudodifferential operators on $\mathbb{R}^n$.

3.A : Introduction

Let (ϕ, φ) be weight functions satisfying (H_1) . Let p be a symbol satisfying the condition (W), i.e :

$p \in S(p; \phi, \varphi)$, and the further condition (S) which is inverse of the condition (N) :

(S) $\exists C, C' > 0, \delta, \delta' > 0$ such that $C p^\delta \leq (\phi\varphi) \leq C' p^{\delta'}$ on $\mathbb{R}^{2n}$.

Let P be $\text{Op}^W p$. It is known that under conditions (W) and (S), P is essentially self-adjoint from $\mathcal{S}(\mathbb{R}^n)$, and semi-bounded from below. If, moreover, (ϕ, φ) satisfies the " ellipticity " condition (H_2) , P has a compact resolvent ([TU-SU], [BE], [RO], [HO 1]).

In the following sections of this paragraph, we will study spectral properties of P , by the mean of a fonctionnal calculus developed according [HE-RO 1] scheme : parametrix, holomorphic fonctionnal calculus, Mellin transform. In this whole paragraph, we suppose that (H_1) , (H_2) , (W) and (S) are true. By possibly translating p , we suppose that P is positive and invertible :

$$P \geq \gamma > 0.$$

3.B : Fonctionnal calculus : parametrix.

For $z \in \mathbb{C} \setminus \mathbb{R}_+$, we have to study $(P-z)^{-1} P^K$ with K positive integer (the reason of the introduction of this parameter K will appear in section 3.C). Thus, we build a parametrix for the equation :

$$(3.1) \quad Q_z^{(K)} \circ (P-z) = P^K.$$

As a first approximation, we get the symbol :

$$(3.2) \quad q_z^{(K)}; 0 = p^K (p-z)^{-1}.$$

Then, by recurrence over j , we define the symbols :

$$(3.3) \quad q_z^{(K)}; j = (p-z)^{-1} [p_j^{(K)} - \sum_{k=0}^{j-1} \sum_{|\alpha+\beta|=j-k} \Gamma(\alpha, \beta) (\partial_\xi^\alpha D_x^\beta p) (\partial_\xi^\beta D_x^\alpha q_z^{(K)}; k)]$$

where $\Gamma(\alpha, \beta) = (\alpha! \beta!)^{-1} 2^{-|\alpha|} (-2)^{-|\beta|}$ and $p_j^{(K)}$ is the j^{th} term in the asymptotics of the symbol of P^K . More precisely, $p_j^{(K)} \in S(p^K (\phi\varphi)^{-j}; \phi, \varphi)$ and is a polynomial expression of the $\partial_\xi^\alpha \partial_x^\beta p$ for $|\alpha+\beta| \leq j$.

In particular, $q_z^{(K)}; 1 = 0$.

By recurrence on $j > 1$, we get :

$$(3.4) \quad q_z^{(K)}; j = \sum_{k=0}^{2j-1} (-1)^k d_{jk}^{(K)} (p-z)^{-k-1}$$

where the $d_{jk}^{(K)}$ are universal polynomial functions of the $\partial_\xi^\alpha \partial_x^\beta p$ for $|\alpha+\beta| \leq j$. $d_{jk}^{(K)}$ belongs

to $S(p^{K+k}(\phi\phi)^{-j}; \phi, \phi)$. For $K = 0$, $d_{jk}^{(0)}$ is simply denoted by d_{jk} (cf [HE-RO 1]).

Weyl's quantization allows the following further details.

(3.5) Lemma

- (1) If $j = 2\ell$, $\ell \in \mathbf{N}^*$, then $d_{jk}^{(K)} = 0$ for each $k \geq 3\ell$
- (2) if $j = 2\ell + 1$, $\ell \in \mathbf{N}$, then $d_{jk}^{(K)} = 0$ for each $k \geq 3\ell + 1$

Proof

Let n_j be the greatest power of $(p-z)^{-1}$ in the expression of $q_z^{(k)}; j$. It is sufficient to show by recurrence on $j \in \mathbf{N}$ that :

$$n_j \leq m_j$$

where $m_{2\ell} = 3\ell + 1$ and $m_{2\ell+1} = 3\ell + 2$.

We have : $n_0 = 1$. On the other hand, $n_1 \leq 1$ because :

$$q_z^{(K)}; 1 = -(1/2)(p-z)^{-1} \{p, (p-z)^{-1}\} + (p-z)^{-1} p_1^{(K)},$$

where $\{ , \}$ is the Poisson's bracket. We note that, in the sum (3.3), the contribution of the term corresponding to $k = j - 1$ (the highest value of k) is :

$$-\frac{1}{2}(p-z)^{-1} \{p, q_z^{(K)}; j-1\}.$$

Now, with (3.4), we are getting :

$$\{p, q_z^{(K)}; j-1\} = \sum_{0 \leq k \leq n_{j-1}-1} (-1)^k \{p, d_{jk}^{(K)}\} (p-z)^{-k-1}$$

As the Poisson's bracket of p with $(p-z)^{-k-1}$ is zero, we find that the greatest power of $(p-z)^{-1}$ in $\{p, q_z^{(K)}; j-1\}$ is equal to n_{j-1} . Then :

$$(3.6) \quad n_j \leq \sup \{n_{j-1} + 1, \max_{k=0}^{j-2} (n_k + j - k + 1)\}.$$

If it is supposed that $n_k \leq m_k$ for $k \leq j - 1$, then (3.6) implies that :

$$n_j \leq \sup \{m_{j-1} + 1, m_{j-2} + 3\}.$$

By looking at the cases when j is even, or odd, we obtain that $n_j \leq m_j$. ■

Let us denote now by $r_z^{(K)}; N$ the symbol of the remainder of the parametrix of (3.1) at the order N , i.e :

$$(3.7) \quad \text{Op}^W(r_z^{(K)}; N) = \text{Op}^W\left(\sum_{j=0}^N q_z^{(K)}; N\right) \circ (P-z) - P^K$$

The composition formula with precised remainder (B.1) allows us to derive a sharp estimate of $r_z^{(K)}; N$.

Let us denote by $d(z)$ the distance in $\mathbf{C}$ between z and the half-axis $[\frac{\gamma}{2}, +\infty[$, where

γ is such that $P \geq \gamma > 0$ (see 3.A).

(3.8) Lemma

$\rho \geq 3/2$ is the number attached to (ϕ, φ) in (B.1). There exists $N_0 > 0$, depending only on p , such that :

for each integers $N \geq 1$ and $K \geq 0$, and each $\alpha, \beta \in \mathbf{N}^n$, there is a $C = C(p, N, K, \alpha, \beta)$ such that we have the estimate on $\mathbf{R}^{2n}$:

$$|\partial_{\xi}^{\alpha} \partial_x^{\beta} r_z^{(K)} ; N| \leq C d(z)^{-1} \left(\frac{|z|}{d(z)} \right)^{\rho N + N_0 + |\alpha| + |\beta|} p^{K+1} (\phi \varphi)^{-N-1} \phi^{-|\alpha|} \varphi^{-|\beta|}$$

for $z \in \mathbf{C} \setminus [\frac{\gamma}{2}, +\infty[$.

Moreover the constant C depends only on a finite number of semi-norms (1.2) of p in $S(p ; \phi, \varphi)$.

Proof :

It follows those of [RO], [HE-RO 1], by using in addition the estimates in (B.1) about the remainder of order $N-j$ in the composition :

$$Op^W(q_z^{(K)} ; j) \circ (P-z) ;$$

for $0 \leq j \leq N$. The number of derivatives on $q_z^{(K)} ; j$ determines by (3.4) the number of powers of $(p-z)^{-1}$, which gives the power of $\frac{|z|}{d(z)}$ in (3.8). (B.1) yields that the number of derivatives on $q_z^{(K)} ; j$ is smaller than $\rho N + N_0$, where N_0 is independant of N .

3.C : Fonctionnal calculus : complex powers.

Under supplementary hypothesis (H_2) on (ϕ, φ) , we are now able to study P^s for $s \in \mathbf{C}$, by the Cauchy's formula : for a suitable path Γ surrounding the half-axis $[\frac{\gamma}{2}, +\infty[$, and for an integer K such that $\text{Re } s < K$, we have :

$$(3.9) \quad P^s = \frac{i}{2\pi} \int_{\Gamma} z^{s-K} p^K \circ (P-z)^{-1} dz .$$

Let :

$$(3.10) \quad p_s ; j = \frac{i}{2\pi} \int_{\Gamma} z^{s-K} q_z^{(K)} ; j dz .$$

The Cauchy's formula and an argument of analytic extension in the s variable, give us that $p_s ; j$ does not depend on Γ or K . Moreover, we have the classical formulas (and thanks to (3.5)) :

$$\begin{aligned} p_s ; 0 &= p^s & p_s ; 1 &= 0 \\ p_s ; j &= \sum_{2 \leq k \leq [3j/2]} \frac{s(s-1)\dots(s-k+1)}{k!} d_{jk} p^{s-k} . \end{aligned}$$

Moreover, we have the following estimates on the remainders :

(3.11) Lemma

We assume that (H_2) is true and we recall that δ is such that $C p^\delta \leq \phi\phi$ (condition (S)).

There exist positive integers N_1 and $N_2 > N_0$ - cf (3.8) - depending only of ϕ, ϕ, p such that for each $L, K : L < 0 < K$ and each $N \geq N_1 + K/\delta$ we have :

- (i) $s \rightarrow P^{S-Op} \sum_{j=0}^N p_s ; j$ is holomorphic in the strip $L \leq \operatorname{Re} s \leq K$, with values in the space of trace class operators on $L^2(\mathbb{R}^n)$.
- (ii) There is $C = C(L, K, N)$ depending only on a finite number of semi-norms of p , such that :

$$\| P^{S-Op} \sum_{j=0}^N p_s ; j \|_{Tr} \leq C(1 + |\operatorname{Im} s|)^{\rho N + N_2}$$

in the strip $L \leq \operatorname{Re} s \leq K$, with ρ as in (B.1) and (3.8).

Proof

With (3.7), (3.9) and (3.10), we get :

$$P^S - Op \sum_{j=0}^N p_s ; j = \frac{i}{2\pi} \int_{\Gamma} z^{s-K} Op r_z^{(K)} ; N \circ (P-z)^{-1} dz .$$

Thanks to (3.8), we derive that $Op r_z^{(K)} ; N$ is of trace class when $p^{(K)}(\phi\phi)^{-N}$ is bounded by $(\phi\phi)^{-N_1}$, for N_1 large enough. With (S), we have :

$$p^K \leq (\phi\phi)^{K/\delta} .$$

Then, we have (i). The point (ii), is deduced from the estimate (3.8) by a method like that of [HE-RO 1] the exponent of $(1 + |\operatorname{Im} s|)$ derives from the exponent of $\frac{|z|}{d(z)}$ in (3.8).

3.D : Fonctionnal calculus : Mellin transform

For a function f in the Schwartz's space $\mathcal{S}(\mathbb{R}^n)$, we may define $f(P)$ as a pseudo-differential operator, the symbol of which is approximated by

$$\sum_{j=0}^N p_f ; j , \text{ where :}$$

$$p_f ; 0 = f(p) ; p_f ; 1 = 0$$

$$p_f ; j = \sum_{2 \leq k \leq [3j/2]} \frac{d_{jk}}{k!} f^{(k)}(p)$$

Here are estimates for the remainder.

(3.12) Proposition

Under the same assumptions than in (3.11), and with the same constants N_1 and

N_2 , we have :

For each $N \geq N_1$, there exists $C = C(p, N)$ depending only on a finite number of semi-norms of p , such that, with $K(N) := [\delta(N - N_1)]$:

$$(3.13) \quad \left\| f(P) - \text{Op}^W \left(\sum_{j=0}^N p_f ; j \right) \right\|_{\text{Tr}} \leq C \left\{ \int_0^{+\infty} \lambda^{-K(N)-1/2} \left| \lambda \partial_\lambda \right|^{\rho N+1+N_2} f_{K(N)}(\lambda) d\lambda + \sum_{k=2}^{\delta N} |f^{(k)}(0)| \right\}$$

where $f_j(\lambda) = f(\lambda) - \sum_{k=0}^j \frac{f^{(k)}(0)}{k!} \lambda^k$ (Taylor's remainder).

Proof

We denote by $M[f] = \int_0^{+\infty} \lambda^{-s} f(\lambda) \frac{d\lambda}{\lambda}$, the Mellin transform of f , which is meromorphic on $\mathbb{C}$, the poles of which coincide with the positive integers. Thanks to the Mellin inversion formula, we get, for any $\sigma_1 < 0$:

$$(3.14) \quad f(P) - \text{Op}^W \sum_{j=0}^N p_f ; j = \frac{1}{2i\pi} \int_{\text{Re } s = \sigma_1} M[f](s) (P^s - \text{Op}^W \sum_{j=0}^N p_s ; j) ds$$

With $\sigma_2 = K(N) + 1/2$, and the Cauchy's formula, we get that (3.14) is equal to

$$(3.14') : (3.14') \quad \frac{1}{2i\pi} \int_{\text{Re } s = \sigma_2} M[f](s) (P^s - \text{Op}^W \sum_{j=0}^N p_s ; j) ds + \sum_{k=0}^{K(N)} (P^k - \text{Op}^W \sum_{j=0}^N p_k ; j) \frac{f^{(k)}(0)}{k!} .$$

Now, thanks to (3.11), we have :

$$\left\| \int_{\text{Re } s = \sigma_2} M[f](s) (P^s - \text{Op}^W \sum_{j=0}^N p_s ; j) ds \right\|_{\text{Tr}} \leq C \int_{\text{Re } s = \sigma_2} |M[f](s)| (1 + |\text{Im } s|)^{\rho N + N_2} d \text{Im } s$$

All that gives (3.12) - see also [HE-RO 1]. ■

3.E : Applications

(3.15) Remark

In the statements (3.11), (3.12), the hypothesis (H_2) allows us to get trace norm estimates for the remainders. If we drop (H_2) , we are still able to estimate those remainders in the scale of weighted Sobolev spaces $H_{\phi, \varphi}^m$ - see [BE].

For instance, if $\varphi = 1$ and $\phi(x, \xi) = (1 + |\xi|^2)^{1/2}$, then $H_{\phi, \varphi}^m$ is the usual Sobolev space $H^m(\mathbb{R}^n)$. If moreover, p is an elliptic symbol of order m , the method of proof of (3.12) shows that, for each $f \in \mathcal{S}(\mathbb{R})$:

$$f(p) \in \bigcap_{k \in \mathbb{N}} S(\phi^{-k} ; \phi, \varphi) \quad (\text{regularizer})$$

We deduce from that, the theorem (1.1) of [KO-SU] namely, the kernel $K(x, y)$, of $f(P)$ is such that :

$$\forall N \in \mathbf{N}, \forall \alpha, \beta \in \mathbf{N}^n, \exists C, |\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C(1+|x-y|)^{-N}, \quad \text{on } \mathbf{R}^{2n} :$$

As a consequence of (3.12), we get, under the same assumptions :

(3.16) Corollary

Let $[a, b]$ be an interval such that $p^{-1}([a, b])$ is a compact set. Then, the spectrum of p contained in $]a, b[$ is discrete.

Proof

As a consequence of the hypothesis and (3.12), $f(p)$ is compact for each $f \in C_0^\infty([a, b])$. That implies (3.16).

(3.17) Corollary (asymptotic expansion)

We assume the same hypotheses than in (3.11) and (3.12) and moreover, that $f \in \mathcal{S}(\mathbf{R})$ and is constant in a neighborhood of 0. Then, we have the following asymptotic expansion of $\text{Tr } f(tp)$ when $t \rightarrow 0_+$:

$$\begin{aligned} \text{Tr } f(tp) &\sim \iint f(tp) \, dx \, d\xi + \\ &\sum_{2 \leq j} \sum_{2 \leq k \leq 3j/2} t^k \iint \frac{d_{jk}}{k!} f^{(k)}(tp) \, dx \, d\xi \end{aligned}$$

Proof

Let f_t be the function defined by $f_t(\lambda) = f(t\lambda)$. It is sufficient to evaluate the behavior of

$$\left| f_t(p) - \sum_{j=0}^N \text{Op}^W p_{f_t}^{(j)} \right| \text{Tr}$$

when $t \rightarrow 0_+$, thanks to (3.12) : in (3.13), the first term is a $O(t^{[\delta(N-N_1)]-1/2})$ and the second term is zero. ■

(3.18) Remarks

(i) There exists $\eta > 0$ such that for all k, j :

$$t^k \iint d_{jk} f^{(k)}(tp) \, dx \, d\xi = O(t^{\delta j - \eta}), \quad t \rightarrow 0_+$$

(ii) If f has some derivatives non-zero in $\lambda = 0$, we still have an asymptotic expansion with the extra-terms :

$$\sum_{k \geq 2} \frac{t^k}{k!} \text{Tr}(p^k + \sum_{\ell=0}^k \text{Op}^W p_k^{(\ell)} f^{(k)}(0))$$

We have showed that the contribution for $k=2$ is zero and that the others are generally non zero.

We can also derive a trace formula for short range Schrödinger operators, which may be compared with the results of [CV] and [GU]. Let $V \in C^\infty(\mathbb{R}^n)$ be the potential, such that there is $s < -n$ and $\theta \in]0, 1]$:

$$(3.19) \quad |\partial_x^\alpha V(x)| \leq C_\alpha (1+|x|)^{s-\theta|\alpha|}.$$

Let us consider quantic hamiltonians $H_0 = -\frac{\Delta}{2}$ and $H = H_0 + V$. For $f \in \mathcal{S}(\mathbb{R})$, constant in a neighborhood of zero, we have :

(3.20) Corollary

For each $t \in \mathbb{R}$, $f(tH) - f(tH_0)$ is trace class and when $t \rightarrow 0$, we have the asymptotics :

$$\text{Tr}(f(tH) - f(tH_0)) \sim \sum_{j \geq 1} t^{j-n/2} C_j(V, f)$$

where, in particular :

$$C_1(V, f) = (2\pi)^{-n} \gamma_n \left(\int_{\mathbb{R}^n} V(x) dx \right) \cdot \left(\int_{\mathbb{R}} (2\rho)^{n/2-1} \cdot f'(\rho) d\rho \right)$$

$$C_2(V, f) = (2\pi)^{-n} \gamma_n \frac{1}{2} \left(\int_{\mathbb{R}^n} V^2(x) dx \right) \cdot \left(\int_{\mathbb{R}} (2\rho)^{n/2-1} \cdot f''(\rho) d\rho \right)$$

$$C_3(V, f) = -(2\pi)^{-n} \gamma_n \frac{1}{6} \left(\int_{\mathbb{R}^n} V^3(x) + \frac{1}{4} |\nabla V(x)|^2 dx \right) \cdot \left(\int_{\mathbb{R}} (2\rho)^{n/2-1} f^{(3)}(\rho) d\rho \right).$$

This statement could be derived from trace formulas in [CV], [GU], when V has a compact support. However, this result seems new for potentials verifying (3.19) only - which ensures that $V \in L^1(\mathbb{R}^n)$.

3 F Weyl's formula

We prove here a uniform estimate of the remainder of Weyl's formula for an hypoelliptic operator $P = \text{Op}^W p$. We denote by $\tilde{N}(\lambda; P)$ and $\tilde{V}(\lambda; p)$ the corresponding counting and volume functions :

$$\tilde{N}(\lambda; P) = \# \{j \in \mathbb{N} \mid \lambda_j(P) \leq \lambda\}$$

$$\tilde{V}(\lambda; p) = (2\pi)^{-n} \int_{p \leq \lambda} dx d\xi.$$

(3.21) Theorem

As in (3.12), we suppose that (H_1) , (H_2) , (W) et (S) are verified for ϕ, φ , and p respectively. Then, for each $\theta \in]0, \frac{\delta}{\rho}]$, there exist C_1, C_2 depending only on a finite number of semi-norms of p , such that, for all $\lambda > 0$:

$$(3.22) \quad |\tilde{N}(\lambda; P) - \tilde{V}(\lambda; p)| \leq (2 + C_1 \lambda^{3\theta-2\delta}) [\tilde{V}(\lambda+\lambda^{1-\theta}; p) - \tilde{V}(\lambda-\lambda^{1-\theta}; p)] + C_2.$$

Proof

We have : $\tilde{N}(\lambda; P) = \text{Tr}(I_{[0, \lambda]}(P))$, and, as in [TU-SU] and [HO 2], we are going to regularize $I_{[0, \lambda]}$. Let $I(\lambda, \theta)$ and $J(\lambda, \theta)$ be the intervals :

$$I(\lambda, \theta) = [\lambda - \lambda^{1-\theta}/2, \lambda + \lambda^{1-\theta}/2]$$

$$J(\lambda, \theta) = [\lambda^{-\lambda^{1-\theta}}, \lambda + \lambda^{1-\theta}].$$

Let $f_{\lambda, \theta}$ and $g_{\lambda, \theta}$ be cut off functions such that :

$$f_{\mu, \theta}(\mu) = \begin{cases} 1 & \text{if } \mu \notin I(\lambda, \theta) \text{ and } \mu < \lambda \\ 0 & \text{if } \mu \notin I(\lambda, \theta) \text{ and } \mu > \lambda \end{cases};$$

$$g_{\lambda, \theta} \equiv C_0^\infty(J(\lambda, \theta)) \text{ and } g_{\lambda, \theta} \equiv 1 \text{ on } I(\lambda, \theta);$$

$$(3.23) \quad \forall k \in \mathbf{N}, |(\mu \partial_\mu)^k f_{\lambda, \theta}(\mu)| + |(\mu \partial_\mu)^k g_{\lambda, \theta}(\mu)| \leq C_k \lambda^{k\theta}.$$

Then, we have :

$$(3.24) \quad |\tilde{N}(\lambda; P) - \text{Tr } f_{\lambda, \theta}(P)| \leq \text{Tr } I_{I(\lambda, \theta)}(P)$$

$$(3.25) \quad \text{Tr } I_{I(\lambda, \theta)}(P) \leq \text{Tr } g_{\lambda, \theta}(P)$$

Now, the first term of $\text{Tr } f_{\lambda, \theta}(P)$ is $\int f_{\lambda, \theta}(p); |\int f_{\lambda, \theta}(p) - \tilde{V}(\lambda; p)|$ is smaller than $\tilde{V}(\lambda + \lambda^{1-\theta}; p) - \tilde{V}(\lambda - \lambda^{1-\theta}; p)$ and so is the first term of $\text{Tr } g_{\lambda, \theta}(P)$.

The j^{th} term of $\text{Tr } f_{\lambda, \theta}(P)$ is bounded by $\lambda^{(3\theta-2\delta)j/2} (\tilde{V}(\lambda + \lambda^{1-\theta}; p) - \tilde{V}(\lambda - \lambda^{1-\theta}; p))$. As the term corresponding to $j=1$ is zero, the highest power of λ is $3\theta-2\delta$ (for $3\theta-2\delta$ is negative !). Thanks to (3.12), we find that the remainder of order N for $\text{Tr } f_{\lambda, \theta}(P)$ is a $O(\lambda^{N(\rho\theta-\delta)+\delta_0})$, and so it is for $\text{Tr } g_{\lambda, \theta}(P)$.

All that, joined to (3.24), (3.25), give (3.22). ■

(3.26) Remarks

If $\phi = \varphi$, we can take $\rho = 3/2$ - see remark (B.2) -. Then, we have roughly the same assumptions that [TU-SU], and, thus, we improve their result which corresponds to $\rho = 2$, and we reach the same limit than [HO 2] : $\theta < 2\delta/3$.

However, under general hypothesis (H_1) , we have but $\rho = 2$, what allows us to reach $\theta < \delta/2$. Nevertheless our estimate is more precise than that of [HO 2], which does not give the behavior in λ of the coefficient multiplying $(\tilde{V}(\lambda + \lambda^{1-\theta}; p) - \tilde{V}(\lambda - \lambda^{1-\theta}; p))$.

§4 : Proof of the main theorem (1.3)

4 A Introduction

The matter consists in proving that :

$$(4.0) \quad N_\pm(\lambda; A) = V_\pm(\lambda; a) + O(\lambda^\zeta V_\pm(\lambda; m))$$

That will result from the two following propositions :

(4.1) Proposition

$$N_\pm(\lambda; A) \leq V_\pm(\lambda; a) + O(\lambda^\zeta V_\pm(\lambda; m))$$

Let us denote :

$N(\lambda ; A) = (N_+ + N_-)(\lambda ; A)$ (counting function of the singular values of A)

$V(\lambda ; A) = (V_+ + V_-)(\lambda ; a)$ (volume function of $|a|$)

(4.2) Proposition

$$N(\lambda ; A) = V(\lambda ; a) + O(\lambda^\zeta V(\lambda ; m))$$

Indeed (4.2) gives that :

$$V_+(\lambda ; a) = N_+(\lambda ; A) + O(\lambda^\zeta V(\lambda ; m)) + N_-(\lambda ; A) - V_-(\lambda ; a)$$

With (4.1) for (-), we get :

$$V_+(\lambda ; a) \leq N_+(\lambda ; A) + O(\lambda^\zeta V(\lambda ; m)),$$

which, with (4.1) for (+), gives (4.0) for (+). ■

In order to prove (4.1) and (4.2), we first need the two basic lemmas :

(4.3) Lemma

For two functions $a, b : \mathbf{R}^{2n} \rightarrow \mathbf{R}$, for $\eta \in [0, 1]$, we have :

$$V_\pm(\lambda ; a+b) \leq V_\pm(\eta \lambda ; a) + V_\pm((1-\eta) \lambda ; b) .$$

Indeed, $a+b \geq \lambda$ implies that $a \geq \eta \lambda$ or $b \geq (1-\eta) \lambda$.

(4.4) Lemma

For two compact self-adjoint operators A, B, we have :

$$N_\pm(\lambda ; A+B) \leq N_\pm(\eta \lambda ; A) + N_\pm((1-\eta) \lambda ; B)$$

It is deduced from Weyl - Ky Fan inequalities for positive or negative eigenvalues, that we will prove in the annex A - because those inequalities are stated only for singular values in [GO-KR].

4 B Preliminary lemmas for weights in the class $S(m ; \phi, \varphi)$

We recall that m verifies the properties (W) , (N), and (T) for its volume function - see § 1.

(4.5) Notation

We denote by a capital letter $(M, B_0, \dots)$ the operator the Weyl symbol of which is the corresponding small letter $(m, b, \dots)$: $M = \text{Op}^W m$, $B_0 = \text{Op}^W b_0, \dots$

(4.6) Lemma

There is $m_1 \in S(m(\phi\varphi)^{-1} ; \phi, \varphi)$ such that, denoting $m_0 = m + m_1$, M_0 is positive and invertible.

Proof

As $m \in S(m ; \phi, \varphi)$, then $m^{-1} \in S(m^{-1} ; \phi, \varphi)$. Thanks to (N), we have $m^{-1} \geq K \phi\varphi$, and

then, $\text{Op}^W m^{-1}$ is semi-bounded. Thus, there is $\mu > 0$, such that $\text{Op}^W(m^{-1} + \mu)$ is strictly positive. Its inverse has its Weyl's symbol like m_0 .

(4.7) Lemma

Let be $\delta \geq 0$ and $d \in S(m^{1+\delta}; \phi, \varphi)$. Then, with the parameters λ_0, γ_1 appearing in condition (T) for m , we have :

$$\forall \lambda \in]0, \lambda_0], N\lambda ; D) \leq C(d) \lambda^\tau V\lambda ; m) \quad \text{with } \tau = \gamma_1 \delta (1+\delta)^{-1},$$

where $C(d)$ depends only on a finite number of semi-norms of d in $S(m^{1+\delta}; \phi, \varphi)$.

(4.8) Convention

That means that $C(d)$ is a polynomial combination of semi-norms $s_k(m^{1+\delta}; \phi, \varphi)$ with coefficients depending only on m, ϕ, φ .

Proof

As in (4.6), there is $\mu > 0$ such that $\text{Op}^W(m^{-1-\delta} + \mu)$ is positive and invertible. Let L be its inverse. As $(m^{-1-\delta} d)$ is bounded like all its derivatives, thanks to composition and Calderón-Vaillancourt's theorems, we get that $L^{-1} \circ D$ is a bounded operator of $L^2(\mathbb{R}^n)$ with a norm $C_1(d)$ depending only on a finite number of semi-norms of d . By mini-max principle, we have so :

$$\lambda_j(D^2) \leq C_1(d) \lambda_j(L^2) \quad \forall j.$$

Therefore :

$$N\lambda ; D) \leq C_1(d) N\lambda ; L).$$

Now, we deduce from theorem (3.21), the rough estimate :

$$N\lambda ; L) \leq C V\lambda ; m^{1+\delta}).$$

From consequence (T') of condition (T), we draw :

$$V\lambda ; m^{1+\delta}) \leq C \lambda^\tau V\lambda ; m).$$

The three last inequalities yield (4.7). ■

Now, we are going to adapt Weyl's formula (3.21) to compact invertible operators.

(4.9) Proposition

Let b be a weight verifying (W) and (N) ; in particular, we have the estimate :

$$(\phi \varphi)^{-1} \leq K(b) b^\gamma$$

and there is $b_1 \in S(b(\phi \varphi)^{-1}; \phi, \varphi)$ such that $B_0 = \text{Op}^W(b + b_1)$ is positive and invertible -see (4.6).

Then, with $\theta_0 = \inf(\frac{\gamma}{2}, \frac{\gamma}{\rho})$, we have for each $0 < \theta \leq \theta_0$ and $\lambda \leq 1/2$:

$$|N(\lambda; B_0) - V(\lambda; b)| \leq (2 + C_1 \lambda^\theta) [V(\lambda - \lambda^{1+\theta}; b) - V(\lambda + \lambda^{1+\theta}; b)] + C_2$$

where C_1 and C_2 depend only on $K(b)$ and a finite number of semi-norms of b and b_1 - see (4.8).

Let us remark that if, moreover, $V(\cdot; b)$ verifies condition (T), we should have :

$$V(\lambda - \lambda^{1+\theta}; b) - V(\lambda + \lambda^{1+\theta}; b) \leq C \lambda^\theta V(\lambda; b).$$

Proof

In order to link (4.9) and (3.21), we denote :

$$P_0 = B_0^{-1}; p = b^{-1}.$$

Then, for $j \geq 1$, there is $p_j \in S(p(\phi\phi)^{-j}; \phi, \phi)$ such that for each $N \geq 1$:

$$P_0 = Op^W(p + \sum_{j=1}^N p_j) = Op^W r_N \quad \text{with} \quad r_N \in S(p(\phi\phi)^{-N-1}; \phi, \phi)$$

and each semi-norm of a p_j depend only on a finite number of semi-norms of b and b_1 - it is the mere construction of a parametrix for B_0 .

Thus, we are in a slightly more general situation than in §3 : the symbol of P_0 is written as an asymptotic expansion (just like classical symbols). Nevertheless, the whole results may be adapted, by starting from :

$$q_z^{(K)}; j = (p-z)^{-1} [p_j^{(K)} - \sum_{k=0}^{j-1} \sum_{\ell=0}^j \sum_{|\alpha+\beta|=j-k-\ell} \Gamma(\alpha, \beta) (\partial_\xi^\alpha D_x^\beta p_\ell) (\partial_\xi^\beta D_x^\alpha q_z^{(K)}; k)]$$

instead of (3.3) (with $p_0 = p$)

Then, instead of (3.22), we arrive to :

$$|\tilde{N}(\lambda; P_0) - \tilde{V}(\lambda, p)| \leq (2 + C_1 \lambda^{\theta-\delta}) [\tilde{V}(\lambda + \lambda^{1-\theta}; p) - \tilde{V}(\lambda - \lambda^{1-\theta}; p)] + C_2$$

with C_1 and C_2 depending only on a finite number of semi-norms of $p, p_1, \dots, p_N$, where N is a fixed integer. Now, let us remark that, for $\theta \leq \theta_0$, $\lambda^{\theta-\delta}$ may be replaced by $\lambda^{-\theta}$, and that, for any fixed constant $c > 0$, $\tilde{V}(\lambda + \lambda^{1-\theta}; p) - \tilde{V}(\lambda - \lambda^{1-\theta}; p)$ may be replaced by $\tilde{V}(\lambda + c\lambda^{1-\theta}; p) - \tilde{V}(\lambda - c\lambda^{1-\theta}; p)$.

On the other hand, we have, for $\mu = \lambda^{-1}$:

$$N(\mu; B_0) = \tilde{N}(\lambda; P_0); V(\mu; b) = \tilde{V}(\lambda; p)$$

and

$$\tilde{V}(\lambda \pm c\lambda^{1-\theta}; p) = V(\mu(1 \pm c\mu^\theta)^{-1}; b)$$

We choose $c = 2^{\theta-1}$, and we have $(1 + c\mu^\theta)^{-1} > 1 - \lambda^\theta$
 $(1 - c\mu^\theta)^{-1} < 1 + \lambda^\theta$.

From all that, we deduce (4.9). ■

Lastly, here is a result looking like Garding's inequality. It consists of a relation between symbols positivity and operators positivity (that relation is not systematic as

it is in anti-wick quantization - see [TU-SU]).

(4.10) Lemma

Let $b \in S(m; \phi, \varphi)$ be a positive symbol. Let be $\epsilon > 0$. Then, there is $d_\epsilon \in S(m(\phi \varphi)^{-1}; \phi, \varphi)$ such that :

$$0 < B + \epsilon M + D_\epsilon$$

and each semi-norm of d_ϵ depends only on a finite number of semi-norms of $(b + \epsilon m)^{1/2}$.

Proof :

$c := (b + \epsilon m)^{1/2} \in S(m^{1/2}; \phi, \varphi)$. There is $d_\epsilon \in S(m(\phi \varphi)^{-1}; \phi, \varphi)$ such that :

$$\text{Op}^W(c^2) + \text{Op}^W d_\epsilon = (\text{Op}^W c)^2.$$

As $(\text{Op}^W c)^2$ is a positive operator, we get (4.10).

4.C Proof of proposition (4.1)

We shall denote by C various constants, independent of involved parameters $(\lambda, \epsilon, \eta, \dots)$; and $C_j(\epsilon)$ for $j=0,1,\dots$, will be positive numbers depending only on ϵ and admitting a polynomial majorization with respect to ϵ^{-1} .

Here is the proof of (4.1) for (+) sign.

Let χ be a C^∞ function on $\mathbf{R}$, $\chi(\lambda) = 0$ if $\lambda < 1$ and $\chi(\lambda) = 1$ if $\lambda > 2$.

For $\epsilon > 0$, we denote, $a_\epsilon = a \chi(a/\epsilon m) + \epsilon m[1 - \chi(a/\epsilon m)]$.

We then have the obvious following properties :

$$(4.11) \quad a_\epsilon \in S(m; \phi, \varphi)$$

$$(4.12) \quad a_\epsilon \geq \epsilon m$$

$$(4.13) \quad a_\epsilon \text{ coïncides with } a \text{ in the region } \{a \geq 2 \epsilon m\}$$

$$(4.14) \quad a \leq a_\epsilon + \epsilon m.$$

Thanks to lemma (4.10), (4.14) gives that there is $d_{1,\epsilon} \in S(m(\phi \varphi)^{-1}; \phi, \varphi)$ such that :

$$A \leq A_\epsilon + 2 \epsilon M + D_{1,\epsilon}.$$

With lemmas (4.6) and (4.10), (4.12) gives that there is $d_{2,\epsilon} \in S(m(\phi \varphi)^{-1}; \phi, \varphi)$ such that :

$$\epsilon M_0/2 \leq A_\epsilon + D_{2,\epsilon}.$$

Let b_ϵ be the summ : $a_\epsilon + d_{2,\epsilon}$. As M_0 is positive and invertible, so it is for B_ϵ . With $d_\epsilon = d_{1,\epsilon} - d_{2,\epsilon}$ we then get :

$$(4.15) \quad A \leq B_\epsilon + 2 \epsilon M + D_\epsilon.$$

We are going to estimate N^+ for each of the summands : at first the error terms $2 \epsilon M + D_\epsilon$ and secondly the principal term B_ϵ ; then we will end the proof.

Step 1 : Error terms

(i) $2 \in M$: Proposition (4.9) yields the rough estimate :

$$N_+(\lambda ; 2 \in M) \leq C V(\lambda ; 2 \in m)$$

with inequality (T') derived from condition (T), we get :

$$V(\lambda ; 2 \in m) = V(\lambda/2\epsilon ; m) \leq (2 \epsilon)^{\gamma_1} V(\lambda ; m) .$$

So, we have :

$$(4.16) N_+(\lambda ; 2 \in M) \leq C \epsilon^{\gamma_1} V(\lambda ; m) .$$

(ii) D_ϵ : as $d_\epsilon \in S(m^{1+\gamma} ; \phi, \phi)$, we draw from lemma (4.7) :

$$(4.17) N(\lambda ; D_\epsilon) \leq C_0(\epsilon) \lambda^\tau V(\lambda ; m) \quad \tau > 0$$

for each semi-norm of d_ϵ is majorized by a polynomial function of ϵ^{-1} .

(iii) $2 \in M + D_\epsilon$: with lemma (4.3) for $\eta = 1/2$, (4.16) and (4.17) give us :

$$N(\lambda ; 2 \in M + D_\epsilon) \leq C(\epsilon^{\gamma_1} + \lambda^\tau C_0(\epsilon)) V(\lambda/2 ; m)$$

and, with the help of inequality (T')

$$(4.18) N(\lambda ; 2 \in M + D_\epsilon) \leq C(\epsilon^{\gamma_1} + \lambda^\tau C_0(\epsilon)) V(\lambda ; m) .$$

Step 2 : Principal term

(i) B_ϵ is positive and invertible, with " main symbol " a_ϵ . As a_ϵ verifies (W) and (N) thanks to (4.11) and (4.12), proposition (4.9) yields :

$$(4.19) N_+(\lambda ; B_\epsilon) \leq V(\lambda ; a_\epsilon) + (2 + C_1(\epsilon) \lambda^\theta) [V(\lambda - \lambda^{1+\theta} ; a_\epsilon) - V(\lambda + \lambda^{1+\theta} ; a_\epsilon)] + C_2(\epsilon) .$$

As the semi-norms of a_ϵ depends on ϵ^{-1} on a polynomial way, so it is for $C_1(\epsilon)$ and $C_2(\epsilon)$.

(ii) We are going to show how $V(\lambda ; a_\epsilon)$ is close to $V_+(\lambda ; a)$.

(4.20) Lemma

$$\gamma_0 = \min(\gamma_1/2, 1/2). \quad \exists C > 0, \forall \lambda < \lambda_0, \forall \epsilon \in]0, 1/2[:$$

$$|V(\lambda ; a_\epsilon) - V_+(\lambda ; a)| \leq C \epsilon^{\gamma_0} V(\lambda ; m)$$

Proof

(1) Majorization of $V(\lambda ; a_\epsilon)$. We have :

$$\int_{a_\epsilon > \lambda} \leq \int_{a > \text{Sup}(\lambda, 2\epsilon m)} + \int_{(a_\epsilon > \lambda) \cap (a < 2\epsilon m)} \\ I_1 \quad I_2$$

Now : $I_1 \leq V_+(\lambda ; a)$ and $I_2 \leq V(\lambda ; 2\epsilon m) \leq C \epsilon^{\gamma_1} V(\lambda ; m)$ by (4.16).

Therefore :

$$V(\lambda ; a_\epsilon) \leq V_+(\lambda ; a) + C \epsilon^{\gamma_1} V(\lambda ; m) .$$

(2) Minoration of $V(\lambda ; a_{\epsilon})$. Thanks to (4.14) we have :

$$V(a_{\epsilon} ; \lambda) \geq V_+(\lambda ; a - \epsilon m) .$$

Thanks to lemma (4.3), we have for $\eta \in [0,1]$:

$$V_+(\lambda ; a - \epsilon m) \geq V_+(\frac{\lambda}{\eta} ; a) - V_+(\frac{1-\eta}{\eta} \lambda ; \epsilon m) .$$

For η such that $\frac{1-\eta}{\eta} = \epsilon^{1/2}$, and thanks to (T') we get :

$$V_+(\frac{\lambda}{\eta} ; a) \geq \eta^{\gamma_2} V_+(\lambda ; a) \geq (1 - \gamma_2 \epsilon^{1/2}) V(\lambda ; a)$$

$$V_+(\frac{1-\eta}{\eta} \lambda ; \epsilon m) \leq \epsilon^{\gamma_1/2} V(\lambda ; m) .$$

So, we obtain the wanted minoration. ■

(iii) Conclusion about B_{ϵ} .

(4.19) and (4.20) give us :

$$N_+(\lambda ; B_{\epsilon}) \leq V_+(\lambda ; a) + C_2(\epsilon) + C[1+C_1(\epsilon) \lambda^{\theta}] [\epsilon^{\gamma_0} V(\lambda ; m) + V_+(\lambda - \lambda^{1+\theta} ; a) - V_+(\lambda + \lambda^{1+\theta} ; a)] .$$

The condition (T) about $V_+(\lambda ; a)$ and the majorization $a \leq C \cdot m$ allow us to deduce :

$$(4.21) \quad N_+(\lambda ; B_{\epsilon}) \leq V_+(\lambda ; a) + C[1+C_1(\epsilon) \lambda^{\theta}] (\epsilon^{\gamma_0} + \lambda^{\theta}) V(\lambda ; m) + C_2(\epsilon) .$$

Step 3 : End of the proof

(4.15) gives us :

$$N_+(\lambda ; A) \leq N_+(\lambda ; B_{\epsilon} + 2 \epsilon M + D_{\epsilon})$$

and with lemma (4.4) :

$$N_+(\lambda ; A) \leq N_+((1-\eta)\lambda ; B_{\epsilon}) + N_+(\eta \lambda ; 2 \epsilon M + D_{\epsilon}) .$$

We take $\eta = \epsilon^{\alpha}$, $\alpha > 0$; then (4.18) and (T') give :

$$N_+(\eta \lambda ; 2 \epsilon M + D_{\epsilon}) \leq C(\epsilon^{\gamma_1} + \lambda^{\tau} \epsilon^{\alpha \tau} C_0(\epsilon)) \epsilon^{-\alpha \gamma_2} V(\lambda ; m) .$$

We then choose α such that $\gamma_1 - \alpha \gamma_2 \equiv \gamma_3 > 0$. So, we get from the last inequalities and

(4.21) :

$$N_+(\lambda ; A) \leq V_+(\lambda ; a) + C(\epsilon^{\gamma_0} + \epsilon^{\gamma_3} + C_1(\epsilon) \lambda^{\theta} + C_3(\epsilon) \lambda^{\tau}) V(\lambda ; m) + C_2(\epsilon) .$$

Now, the inequality $K m^{\gamma} \geq (\phi \phi)^{-1}$ implies that there is some $\tau' > 0$ such that

$$V(\lambda ; m) \geq C \lambda^{-\tau'} \quad (\text{with } C > 0)$$

Therefore, with $\omega = \inf(\theta, \tau, \tau')$, $C_4 = C_1 + C_2 + C_3$ and $\gamma_4 = \inf(\gamma_0, \gamma_3)$, we get :

$$N_+(\lambda ; A) \leq V_+(\lambda ; a) + C(\epsilon^{\gamma_4} + C_4(\epsilon) \lambda^{\omega}) V(\lambda ; m) .$$

Now, there is some positive integer N such that $C_4(\epsilon) \leq \epsilon^{-N}$ for $\epsilon < 0$ small enough.

Then, we choose :

$$\epsilon = \lambda^{\beta} \text{ with } \beta = \frac{\omega}{\gamma_4 + N}$$

Thus, we get the proposition (4.1) with $\zeta = \gamma_4 \beta$.

4.D Proof of proposition (4.2)

In view of (4.1), it remains to be proved:

$$(4.22) \quad V(\lambda; a) \leq N(\lambda; A) + O(\lambda^{\zeta} V(\lambda; m)).$$

Step 1 : Reduction to the case when a is positive

Let us admit that (4.22) is true for $a \geq 0$. Then, for any symbol $a \in S(m; \phi, \varphi)$, a^2 is positive and belongs to $S(m^2; \phi, \varphi)$. We get from (4.22) :

$$(4.23) \quad V(\lambda; a^2) \leq N(\lambda; \text{Op}^W a^2) + O(\lambda^{\zeta'} V(\lambda; m^2)).$$

Now there is $d \in S(m^2(\phi\varphi)^{-1}; \phi, \varphi)$ such that :

$$\text{Op}^W a^2 - A^2 = \text{Op}^W d.$$

Thanks to lemmas (4.4) and (4.7), we deduce from (4.23) that, by possibly decreasing

ζ' :

$$(4.24) \quad V(\lambda; a^2) \leq N(\lambda; A^2) + O(\lambda^{\zeta'} V(\lambda; m^2)).$$

As $N(\lambda; A^2) = N(\lambda^{1/2}; A)$ and $V(\lambda; a^2) = V(\lambda^{1/2}; a)$, we deduce (4.22) from (4.24) with $\zeta = 2\zeta'$.

Step 2 : Proof of (4.22) in the case when a is positive

Let a_ϵ be $a + \epsilon m$. Thanks to lemmas (4.6) and (4.10) we obtain that there is $d_\epsilon \in S(m(\phi\varphi)^{-1}; \phi, \varphi)$ such that :

$$B_\epsilon \equiv A_\epsilon + D_\epsilon \geq \epsilon M_0/2.$$

We apply to B_ϵ with main symbol a_ϵ , the proposition (4.9) and we get :

$$(4.25) \quad V(\lambda; a_\epsilon) \leq N(\lambda; B_\epsilon) + (1 + C_1(\epsilon) \lambda^\theta) (V(\lambda - \lambda^{1+\theta}; a_\epsilon) - V(\lambda + \lambda^{1+\theta}; a_\epsilon)) + C_2(\epsilon).$$

Thanks to lemma (4.3) and the condition (T) for $V(., a)$, we obtain a similar statement to (4.20). At last :

$$(4.26) \quad N(\lambda; B_\epsilon) \leq N((1-\eta)\lambda; A) + N(\eta\lambda; \epsilon M + D_\epsilon).$$

We use (4.26) for $\eta = \epsilon^\alpha$ with a suitable $\alpha > 0$ and we transfer the obtained estimate to (4.25) ; we then get (4.2) like (4.1). ■

(4.27) Remark

For the minoration of $N(\lambda)$ it is possible to reduce to the case when a is positive and to approximate a **everywhere** by an elliptic regularization (and to get (4.26)). That would be impossible for $N_+(\lambda)$ alone when a is not of constant sign (a_ϵ in 4.C is far from a when a is negative).

Annex A : Weyl-Ky Fan inequalities

In order to prove these inequalities, we start giving a Ky Fan's type characterization for the eigenvalues of a compact self-adjoint operator T in a Hilbert space H . We will deduce Weyl's inequalities for eigenvalues, then for counting functions $N_{\pm}(\lambda)$.

With the same convention as in §1 about the eigenvalues $\lambda_j^{\pm}(T)$ of T , we state:

(A.1) Proposition

$$\lambda_j^+(T) = \inf_{K,S} \|T-K-S\|,$$

where the inf is computed for all self-adjoint operators K of rank $< j$, and for all compact non positive operators S .

$$\lambda_j^-(T) = -\inf_{K,S} \|T-K-S\|,$$

with S non negative.

[GO-KR] give the corresponding statement for singular values : $s_j(T) = \inf_K \|T-K\|$, with K of rank $< j$.

Proof

We study the (+) sign. We denote by (φ_k^+) , (φ_k^-) an orthonormal eigenfunctions basis of H , associated to the (λ_k^+) , (λ_k^-) .

$$(i) \quad \text{As :} \quad T = \sum_{k \geq 1} \lambda_k^+ \varphi_k^+ \otimes \varphi_k^+ + \sum_{k \geq 1} \lambda_k^- \varphi_k^- \otimes \varphi_k^-$$

by taking :

$$K = \sum_{k=1}^{j-1} \lambda_k^+ \varphi_k^+ \otimes \varphi_k^+, \quad S = \sum_{k \geq 1} \lambda_k^- \varphi_k^- \otimes \varphi_k^-$$

we have suitable K and S , and we get : $\|T-K-S\| = \lambda_j^+$. So, $\lambda_j^+ \geq \inf_{K,S} \|T-K-S\|$.

(ii) If j is such that φ_j^+ does not exist (if the positive part of T is of rank $< j$), we have $\inf_{K,S} \|T-K-S\| = 0$, and thus $\inf_{K,S} \|T-K-S\| = \lambda_j^+ = 0$.

If φ_j^+ exists, let K and S be self-adjoint operators : K of rank $< j$ and S negative. And let $u \neq 0$ be orthogonal to $\text{Rg}(K)$ in the space with the basis $\varphi_1^+, \dots, \varphi_j^+$. We have :

$$\begin{aligned} \|(T-K-S)u\| &\geq \|(T-S-K)u, u\rangle \\ &\geq \langle Tu, u \rangle \text{ for } \langle Ku, u \rangle = 0 \text{ and } \langle -Su, u \rangle \geq 0 \\ &= \sum_{k=1}^j \lambda_k^+ \langle u, \varphi_k^+ \rangle^2 \geq \lambda_j^+ \|u\|^2. \end{aligned}$$

Therefore $\|T-K-S\| \geq \lambda_j^+$.

(A.2) Corollary

$$\lambda_{j+k+1}^+(T_1+T_2) \leq \lambda_{j+1}^+(T_1) + \lambda_{k+1}^+(T_2).$$

Proof

Let K_1, K_2, S_1, S_2 be self-adjoint operators such that, $\text{rank}(K_1) \leq j$, $\text{rank}(K_2) \leq k$, $S_1, S_2 \leq 0$, and :

$$\|T_1 - K_1 - S_1\| = \lambda_{j+1}^+(T_1); \|T_2 - K_2 - S_2\| = \lambda_{k+1}^+(T_2).$$

As, $\text{rank}(K_1 + K_2) \leq j + k$ and $S_1 + S_2 \leq 0$, we get (A.2). ■

In order to prove lemma (4.4), it remains to state :

(A.3) Proposition

Let (u_n) , (v_n) , (w_n) be three decreasing sequences of positive numbers, tending to zero. We denote :

$N(\lambda; u) = \# \{n \geq 1 / u_n \geq \lambda\}$, and the same for v, w .

Then (i) implies (ii) :

$$(i) \quad j, k \in \mathbf{N} : u_{j+k+1} \leq v_{j+1} + w_{k+1}$$

$$(ii) \quad \lambda < 0, \quad \eta \in [0, 1], \quad N(\lambda; u) \leq N(\eta\lambda; v) + N((1-\eta)\lambda; w)$$

Indeed, (i) is equivalent to (ii).

Proof

Let $\lambda > 0$ and $\eta \in [0, 1]$ be real numbers.

a) if $v_1 < \eta\lambda$ and $w_1 < (1-\eta)\lambda$, then :

$$u_1 \leq v_1 + w_1 < \lambda.$$

Thus $N(\lambda; u) = 0$ and the inequality (ii) is true for λ, η .

b) If not, let us suppose that $v_1 \geq \eta\lambda$. Let j be the integer such that : $u_{j+1} < \lambda \leq u_j$ (if j does not exist, $N(\lambda; u) = 0$, and (ii) is true). We have : $N(\lambda; u) = j$.

b1) if $v_j \geq \eta\lambda$, then $N(\eta\lambda; v) \geq j$ and (ii) is true.

b2) if not : $v_j < \eta\lambda$. Let $k < j$ be the integer such that $v_{k+1} < \eta\lambda \leq v_k$ (k exists because $v_1 \geq \eta\lambda$). We have : $N(\eta\lambda; v) = k$.

Now, thanks to (i) :

$$u_j \leq v_{k+1} + w_{j-k}$$

As $u_j \geq \lambda$ and $v_{k+1} < \eta\lambda$, we get :

$$w_{j-k} \geq \lambda(1-\eta).$$

So : $N((1-\eta)\lambda; w) \geq j-k$.

Therefore :

$$N(\lambda; u) = j = k + (j-k) \leq N(\eta\lambda; v) + N((1-\eta)\lambda; w). \quad \blacksquare$$

In order to deduce lemma (4.4), we apply (A.3) to the sequences $u = (\lambda_j^+(A+B))$, $v = (\lambda_j^+(A))$, $w = (\lambda_j^+(B))$, in basing our argument on corollary (A.2) to have (i).

[BI-SO 2] gives without proof the statement corresponding to lemma (4.4) for the singular values.

Annexe B : Composition formula with precised remainder

Let (ϕ, φ) be weight functions verifying (H_1) . Let, for $i = 1, 2$, m_i be (ϕ, φ) -temperate weights and $a_i \in S(m_i; \phi, \varphi)$. It is well known that the operator $(Op^W a_1) \circ (Op^W a_2)$ admit a Weyl's symbol a given by :

$$(B.0) \quad a(x, \xi) = \exp \left\{ \frac{i}{2} \sigma(D_z, D_\zeta; D_y, D_\eta) \right\} a_1(z, \zeta) a_2(y, \eta) \quad \left| \begin{array}{l} z=y=x \\ \zeta=\eta=\xi \end{array} \right.$$

where σ is the symplectic form :

$$\sigma(z, \zeta; y, \eta) = \langle \zeta, y \rangle - \langle z, \eta \rangle.$$

With the help of Taylor's formula, we determine the principal part a_N of (B.0) at the order N , for each integer N :

$$a_N(x, \xi) = \sum_{j=0}^N \frac{1}{j!} \left(\frac{i}{2} \sigma(D_z, D_\zeta; D_y, D_\eta) \right)^j a_1(z, \zeta) a_2(y, \eta) \quad \left| \begin{array}{l} z=y=x \\ \zeta=\eta=\xi \end{array} \right.$$

and the remainder r_N at the order N :

$$r_N(a_1, a_2; x, \xi) = a(x, \xi) - a_N(x, \xi).$$

The aim of this annex is to state :

(B.1) Proposition : universal estimate of the remainder.

There exist $\rho \geq 3/2$, $\rho' \geq 3/2$ depending only on (ϕ, φ) such that for each $N \geq 1$ and $q \geq 0$, there is $\gamma = \gamma(N, q, n)$ and $j(n)$ such that for all $(x, \xi) \in \mathbb{R}^{2n}$:

$$\sum_{|\alpha|+|\beta| \leq q} \left| \partial_\xi^\alpha \partial_x^\beta r_N(a_1, a_2)(x, \xi) \right| \leq$$

$$\gamma(q, N, n) \left(\sum_{\substack{N < k < \rho N + M_2 + j(n) + q \\ N < \ell < \rho' N + M_1 + j(n) + q + M}} s_k(a_1; m_1) s_\ell(a_2; m_2) \right) m_1(x, \xi) m_2(x, \xi) (\phi\varphi)^{-N}(x, \xi)$$

where M_1 is the exponent attributed to m_1 in the definition (1.1/2) of a temperate weight, and M depends only on ϕ, φ, M_1 and M_2 .

(B.2) Remarks

(1) Under our hypothesis (H_1) , we will show in the proof that we may take $\rho = 2$.

(2) If, moreover, $\phi = \varphi$, then we may take $\rho = 3/2$.

(3) We conjecture that (B.1) is still true with the relaxed hypothesis (H'_1) :

$$(H'_1) \quad \left\{ \begin{array}{l} \phi\varphi \geq 1 \text{ on } \mathbb{R}^{2n} \\ \phi \text{ and } \varphi \text{ are } (\phi, \varphi)\text{-temperate} \end{array} \right.$$

Proof

The inversion Fourier's formula, and the Taylor's formula with integral remainder, give :

$$(B.3) \quad r_{N,t}(a_1, a_2; x, \xi) = \frac{1}{N!} \int_0^1 (1-t)^{N-1} r_{N,t}(a_1, a_2; x, \xi) dt$$

with $r_{N,t}$ defined by :

$$(B.4) \quad r_{N,t}(a_1, a_2, x, \xi) = \int e^{i \frac{t}{2} \sigma(\tilde{z}, \tilde{\zeta}; \tilde{y}, \tilde{\eta})} \sigma(\tilde{z}, \tilde{\zeta}; \tilde{y}, \tilde{\eta})^N e^{i(\tilde{z}x + \tilde{\zeta}\xi + \tilde{y}x + \tilde{\eta}\xi)} \hat{a}_1(\tilde{z}, \tilde{\zeta}) \hat{a}_2(\tilde{y}, \tilde{\eta}) d\tilde{z} d\tilde{\zeta} d\tilde{y} d\tilde{\eta}$$

where $\hat{a}$ denotes the total Fourier's transform of a , and $d\tilde{z} = (2\pi)^{-n} dz$.

Thanks to the well-known result about Fourier's transform of gaussian functions :

$$\mathcal{F} \left(e^{i \frac{t}{2} \sigma(\tilde{z}, \tilde{\zeta}; \tilde{y}, \tilde{\eta})} \right) (z, \zeta; y, \eta) = \left(\frac{2\pi}{t} \right)^{2n} e^{-\frac{i}{2t} \sigma(z, \zeta; y, \eta)}$$

and thanks to Plancherel's theorem, (B.4) yields for $t \neq 0$:

$$(B.5) \quad r_{N,t}(a_1, a_2; x, \xi) = (2\pi t)^{-2n} \int e^{-\frac{i}{2t} \sigma(z, \zeta; y, \eta)} \sigma^N(D_z, D_\zeta; D_y, D_\eta) a_1(z+x, \zeta+\xi) a_2(y+x, \eta+\xi) dz d\zeta dy d\eta$$

We obviously have :

$$r_{N,0}(a_1, a_2; x, \xi) = \sigma^N(D_z, D_\zeta; D_y, D_\eta) a_1(z, \zeta) \cdot a_2(y, \eta) \Big|_{\substack{z=y=x \\ \zeta=\eta=\xi}}$$

We are going to study the expression (B.5) of $r_{N,t}$, by a classical method consisting in isolating the critical point 0 of the phase. Let $\chi \in C_0^\infty([-2, 2])$ be a cut off function, $\chi \equiv 1$ on $[-1, 1]$. Let be $\varepsilon > 0$ be a small parameter. We denote :

$$b(z, \zeta, y, \eta; x, \xi) = \sigma^N(D_z, D_\zeta; D_y, D_\eta) a_1(z+x, \zeta+\xi) a_2(y+x, \eta+\xi) \text{ also denoted } \sigma^N(a_1, a_2)$$

$$b_1(z, \zeta, y, \eta; x, \xi) = \chi \left(\frac{|y|^2 + |z|^2}{\varepsilon^2 \varphi^2(x, \xi)} + \frac{|\eta|^2 + |\zeta|^2}{\varepsilon^2 \phi^2(x, \xi)} \right) b(z, \zeta, y, \eta; x, \xi) \text{ and } b_2 = b - b_1.$$

We are going to estimate the contribution of b_1 to (B.5) - lemma (B.6) - and then the contribution of b_2 - lemma (B.7). By integrating in $t \in [0, 1]$, we shall get (B.1).

(B.6) Lemma

For $\varepsilon > 0$ small enough, there exists $\gamma_1(N, n) > 0$ and $j_1(n) \in \mathbf{N}$ such that for all $a_i \in S(m_i; \phi, \varphi)$, $t \in [0, 1]$ and $(x, \xi) \in \mathbf{R}^{2n}$:

$$|t^{-2N} \int e^{-\frac{i}{2t} \sigma(z, \zeta; y, \eta)} b_1(z, \zeta, y, \eta; x, \xi) dz d\zeta dy d\eta| \leq$$

$$t^{-2n/(2n+1)} \gamma_1(N, n) \left(\sum_{\substack{N \leq k \leq N+j_1(n) \\ N \leq \ell \leq N+j_1(n)}} s_k(a_1; m_1) s_\ell(a_2; m_2) \right) m_1(x, \xi) m_2(x, \xi) \cdot (\phi \varphi)^{-N}(x, \xi).$$

Proof

In b_1 support, we have, thanks to the (ϕ, φ) -continuity :

$$\phi^{-1}(x+z, \xi+\zeta) \leq \phi^{-1}(x, \xi) \quad \text{and} \quad \phi^{-1}(x+y, \xi+\eta) \leq \phi^{-1}(x, \xi)$$

and so it is for ϕ^{-1} , m_1 and m_2 .

We cut out again b_1 support to control the behavior with respect to t : $b_1 = b'_1 + b''_1$ with b'_1 near to the critical point and b''_1 far from the critical point. Precisely, with $\varepsilon_0 = 1/(2n+1)$, we define :

$$b'_1(z, \zeta, y, \eta; x, \xi) = \chi\left(\frac{|z|^2 + |\zeta|^2 + |y|^2 + |\eta|^2}{1 - \varepsilon_0}\right) b_1(z, \zeta, y, \eta; x, \xi) \quad \text{and} \quad b''_1 = b_1 - b'_1.$$

The volume of b'_1 support is a $O(t^{2n(1-\varepsilon_0)})$. That fact allows us to estimate b'_1 contribution.

We estimate b''_1 contribution by integration by parts $2n(2n+1)$ -times with the differential operator :

$$L = \frac{2 \, it}{|z|^2 + |\zeta|^2 + |y|^2 + |\eta|^2} \left(\zeta \cdot \frac{\partial}{\partial y} - \eta \cdot \frac{\partial}{\partial z} + y \cdot \frac{\partial}{\partial \zeta} - z \cdot \frac{\partial}{\partial \eta} \right).$$

Indeed, we observe that each integration with L allows a gain of t^{ε_0} , and that the volume of b''_1 support is a $O((\phi \varphi)^{2n}(x, \xi))$. By developping the composition formula up to $N+2n+1$, we get the wanted estimate about r_N , which is given by :

$$r_N = \sum_{j=N+1}^{N+2n+1} a_j + r_{N+2n+1}.$$

(B.7) Lemma

There exists $\gamma_2(N, n) > 0$ and $j_2(n) \in \mathbf{N}$, such that for all $a_1 \in S(m_1; \phi, \varphi)$, $t \in]0, 1]$ and $(x, \xi) \in \mathbf{R}^{2n}$:

$$|t^{-2n} \cdot \int e^{-\frac{i}{2t} \sigma(z, \zeta; y, \eta)} b_2(z, \zeta, y, \eta; x, \xi) \, dz \, d\zeta \, dy \, d\eta| \leq$$

$$\gamma_2(N, n) \left(\sum_{\substack{N \leq k \leq 2N+M_2 + j_2(n) \\ N \leq l \leq 2N+M_1 + j_2(n)}} s_k(a_1; m_1) s_l(a_2; m_2) \right) m_1(x, \xi) m_2(x, \xi) (\phi \varphi)^{-N}(x, \xi).$$

Proof

On b_2 support, we have

$$\frac{|y|^2 + |\zeta|^2}{\varphi^2(x, \xi)} + \frac{|\eta|^2 + |z|^2}{\phi^2(x, \xi)} \geq \varepsilon^2.$$

We divide that region into three zones Z_1, Z_2, Z_3 defined by :

$$Z_1 : \frac{|z|^2}{\varphi^2(x, \xi)} + \frac{|\zeta|^2}{\phi^2(x, \xi)} \leq \frac{\varepsilon^2}{2} \quad \text{and} \quad \frac{|y|^2}{\varphi^2(x, \xi)} + \frac{|\eta|^2}{\phi^2(x, \xi)} \geq \frac{\varepsilon^2}{2}$$

$$Z_2 : \frac{|z|^2}{\varphi^2(x, \xi)} + \frac{|\zeta|^2}{\phi^2(x, \xi)} \geq \frac{\varepsilon^2}{2} \quad \frac{|y|^2}{\varphi^2(x, \xi)} + \frac{|\eta|^2}{\phi^2(x, \xi)} \leq \frac{\varepsilon^2}{2}$$

$$Z_3 : \frac{|z|^2}{\phi^2(x,\xi)} + \frac{|\zeta|^2}{\phi^2(x,\xi)} \geq \frac{\varepsilon^2}{2} \quad \frac{|y|^2}{\phi^2(x,\xi)} + \frac{|\eta|^2}{\phi^2(x,\xi)} \geq \frac{\varepsilon^2}{2}$$

We deduce from that partition, a breaking up of b_2 into three parts, $b_2^{(j)}$, the support of which being close to Z_j for $j=1,2,3$.

At first we are going to study $b_2^{(1)}$ in lemma (B.8). We note that $b_2^{(2)}$ is like $b_2^{(1)}$. At last, we will study $b_2^{(3)}$ in lemma (B.10).

(B.8) Lemma

For all $a_i \in S(m_i; \phi, \varphi)$, for all $t \in [0,1]$, $(x,\xi) \in \mathbb{R}^{2n}$, we have :

$$|t^{-2n} \int e^{-\frac{i}{2t} \sigma(z,\zeta; y,\eta)} b_2^{(1)}(z,\zeta,y,\eta; x,\xi) dz d\zeta dy d\eta| \leq$$

$$\gamma(N,n) \sum_{N \leq k \leq 2N + M_2 + j(n)} s_k(a_1; m_1) s_N(a_2; m_2) m_1(x,\xi) m_2(x,\xi) (\phi \varphi)^{-N}(x,\xi).$$

Proof :

We integrate by parts with the help of operator L_1 :

$$(B.9) \quad L_1 = \frac{2it}{|y|^2 \phi^2(x,\xi) + |\eta|^2 \varphi^2(x,\xi)} \left(\phi^2(x,\xi) y \frac{\partial}{\partial \zeta} - \varphi^2(x,\xi) \eta \frac{\partial}{\partial z} \right).$$

Its acts on $a_1(x+z, \xi+\zeta)$. The condition about (z,ζ) which is imposed in the zone Z_1 , implies that L_1 allows a gain consisting of

$$t(1+|y|^2 \phi^2(x,\xi) + |\eta|^2 \varphi^2(x,\xi))^{-1/2}$$

or :

$$t(\phi \varphi)^{-1}(x,\xi) \text{ (thanks to condition about } (y,\eta)).$$

We integrate $(2n+1)$ -times to obtain the convergence of the integral with respect to (y,η) ; then we again integrate N -times to gain $(\phi \varphi)^{-N}(x,\xi)$. Here, we use the boundedness of ϕ^{-1} and φ^{-1} on $\mathbb{R}^{2n}$, in order to estimate $\sigma^N(a_1, a_2)$ - hypothesis (H_1) - . Thus we get (B.8).

(B.10) Lemma

There exists M_0 depending only on ϕ and φ , and $\gamma(n,N)$ such that :

$$|t^{-2n} \int e^{-\frac{i}{2t} \sigma(z,\zeta; y,\eta)} b_2^{(3)}(z,\zeta,y,\eta; x,\xi) dz d\zeta dy d\eta| \leq$$

$$\gamma(N,n) \sum_{N \leq k \leq 2N + M_2 + 2n + 1} s_k(a_1; m_1) s_l(a_2; m_2) m_1(x,\xi) m_2(x,\xi) (\phi \varphi)^{-N}(x,\xi).$$

$$N \leq l \leq (M_0+1)N + M_0(M_2+2n+1) + M_1 + 2n + 1$$

Proof

In that zone Z_3 , we do use that ϕ^{-1} and φ^{-1} are $(1,1)$ temperate. So, it exists $C, M_0 > 0$

such that :

$$\phi^{-1}(x+z, \xi+\zeta) \leq \phi^{-1}(x, \xi) (1+|z|+|\zeta|)^{M_0}$$

$$\phi^{-1}(x+z, \xi+\zeta) \leq \phi^{-1}(x, \xi) (1+|z|+|\zeta|)^{M_0}.$$

Thus, each integration with the above operator L_1 - (B.9) - allows a gain consisting of :

$$t(1+|y|^2 \phi^2(x, \xi) + |\eta|^2 \varphi^2(x, \xi))^{-1/2} (1+|z|+|\zeta|)^{M_0}$$

or :

$$t(\phi \varphi)^{-1}(x, \xi) (1+|z|+|\zeta|)^{M_0/2}.$$

Thus we integrate by parts $(N+M_2+2n+1)$ -times with the help of L_1 . But we have a loss with respect to (z, ζ) variable. To overcome that, we integrate

$$[M_0(N+M_2+2n+1) + M_1 + 2n + 1] - \text{times}$$

with the help of operator L_2 :

$$L_2 = \frac{2it}{|z|^2+|\zeta|^2} (z \frac{\partial}{\partial \eta} - \zeta \frac{\partial}{\partial y}).$$

Indeed, thanks to (H_1) , L_2 allows a gain on Z_3 consisting of $(1+|z|^2+|\zeta|^2)^{-1/2}$. Thus we get (B.10).

Proof of remark (B.2/2)

Now, we suppose moreover that $\phi = \varphi$. We read again (B.8) and (B.10) proofs.

Lemma (B.8) : as $\sigma^N(a_1, a_2)$ is majorized by $\gamma m_2 s_N(a_2 ; m_2) [m_1 s_N(a_1 ; m_1) (\phi \varphi)^{-\frac{N}{2}}]$ $[N/2]+2n+2$ integrations by parts with L_1 are enough. So, in the estimate (B.8), we have $N \leq k \leq 3N/2 + M_2 + j(n)$.

Lemma (B.10) : here, we have the majorization :

$$|\sigma^N(a_1, a_2)(z, \zeta, y, \eta ; x, \xi)| \leq \gamma m_2 s_N(a_2 ; m_2) \cdot$$

$$[m_1 s_N(a_1 ; m_1) (\phi \varphi)^{-N/2}(x, \xi) (1+|z|+|\zeta|)^{M_0 N}].$$

Thus, we integrate $([N/2] + M_2 + 2n + 2)$ -times with L_1 to have a suitable behavior with respect to (y, η) variables, and :

$$M_0([3N/2] + M_2 + 2n + 2) + M_1 + 2n + 1 - \text{times}$$

with L_2 for the behavior in (z, ζ) variables. Thus, in the estimate (B.10), we may sum on :

$$N \leq k \leq 3N/2 + M_2 + j(n)$$

$$N \leq \ell \leq (3M_0/2 + 1) N + M_1 + j(n) + M$$

where M depends only on m_1, m_2, ϕ and φ .

It is clear that those modifications yield the proof of (B.2/2).

Annex C : Computation for an example

We are going to compute the volume function of a symbol $a(x, \xi) = b(x) c(\xi)$ where b and c are C^∞ and homogeneous of order $\sigma < 0$, outside the ball $B(0, 1/2)$ in $\mathbb{R}^n$. We have :

$$(C.1) \quad V(a; \lambda) = \int_{a \geq \lambda} dx d\xi = V_1(a; \lambda) + V_2(a; \lambda)$$

where for $i=1, 2$:

$$V_i(a; \lambda) = \int_{J_i} dx d\xi$$

with :

$$J_1 = \{(x, \xi) \in \mathbb{R}^{2n} / (|x| \leq 1 \text{ or } |\xi| \leq 1) \text{ and } a(x, \xi) \geq \lambda\}$$

$$J_2 = \{(x, \xi) \in \mathbb{R}^{2n} / (|x| \geq 1 \text{ and } |\xi| \geq 1) \text{ and } a(x, \xi) \geq \lambda\}$$

V_1 may be computed just like the volume function of a symbol with one predominant homogeneity. So :

$$(C.2) \quad V_1(a; \lambda) = \frac{\lambda^{n/\sigma}}{n} \int_{B^n \times S^{n-1}} b(x)^{-n/\sigma} c(\tilde{\xi})^{-n/\sigma} dx d\tilde{\xi} \\ + \frac{\lambda^{n/\sigma}}{n} \int_{S^{n-1} \times B^n} c(\tilde{\xi})^{-n/\sigma} b(\tilde{x})^{-n/\sigma} d\tilde{\xi} d\tilde{x} + o(1) .$$

We have denoted :

$$\tilde{x} = \frac{x}{|x|}, \quad \tilde{\xi} = \frac{\xi}{|\xi|}, \quad B^n = \{x \in \mathbb{R}^n, |x| \leq 1\} \quad \text{or} \quad \{\xi \in \mathbb{R}^n, |\xi| \leq 1\} .$$

Let us compute V_2 :

$$V_2(a; \lambda) = \int_{\substack{|x|^\sigma b(\tilde{x}) |\xi|^\sigma c(\tilde{\xi}) \geq \lambda, \\ |x| \geq 1, |\xi| \geq 1}} |x|^{n-1} |\xi|^{n-1} d|x| d|\xi| d\tilde{x} d\tilde{\xi} \\ = \int_{\substack{|x|^\sigma b(\tilde{x}) c(\tilde{\xi}) \geq \lambda, \\ |x| \geq 1}} |x|^{n-1} v(x, \tilde{\xi}) d|x| d\tilde{x} d\tilde{\xi}$$

where :

$$v(x, \tilde{\xi}) = \int_1^{\lambda b^{-1}(\tilde{x}) c^{-1}(\tilde{\xi})^{1/\sigma}} |x|^{-1} r^{n-1} dr \\ = \frac{1}{n} \lambda^{n/\sigma} b(\tilde{x})^{-n/\sigma} c(\tilde{\xi})^{-n/\sigma} |x|^{-n} - \frac{1}{n} .$$

Thus :

$$V_2(a; \lambda) = \int_{b(\tilde{x}) c(\tilde{\xi}) \geq \lambda} w(\tilde{x}, \tilde{\xi}) d\tilde{x} d\tilde{\xi}$$

where :

$$w(\tilde{x}, \tilde{\xi}) = \int_1^{\lambda b^{-1}(\tilde{x}) c^{-1}(\tilde{\xi})^{1/\sigma}} \frac{1}{n} \lambda^{n/\sigma} b^{-n/\sigma}(\tilde{x}) c^{-n/\sigma}(\tilde{\xi}) \frac{1}{r} - \frac{r^{n-1}}{n} dr .$$

We have :

$$w = \frac{1}{n\sigma} \lambda^{n/\sigma} (bc)^{-n/\sigma} \text{Log} \frac{\lambda}{bc} - \frac{\lambda^{n/\sigma}}{n^2} (bc)^{-n/\sigma} + \frac{1}{n^2}$$

Thus :

$$V_2(a; \lambda) = \frac{\lambda^{n/\sigma}}{n^2} \int_{bc > \lambda} (bc)^{-n/\sigma} \left[\frac{n}{\sigma} \log \lambda + \log(bc)^{-n/\sigma} - 1 \right] d\tilde{x} d\tilde{\xi} + O(1)$$

Now :

$$\int_{bc \leq \lambda} (bc)^{-n/\sigma} \log(bc)^{-n/\sigma} d\tilde{x} d\tilde{\xi} = O(\lambda^{-n/\sigma} \log \frac{1}{\lambda}) .$$

Therefore, we have :

$$\begin{aligned} \text{(C.3)} \quad V_2(a; \lambda) &= \frac{\lambda^{n/\sigma}}{n\sigma} \log \lambda \int_{S^{n-1} \times S^{n-1}} (bc)^{-n/\sigma} d\tilde{x} d\tilde{\xi} \\ &+ \frac{\lambda^{n/\sigma}}{n^2} \int_{S^{n-1} \times S^{n-1}} (bc)^{-n/\sigma} \log \frac{(bc)^{-n/\sigma}}{e} d\tilde{x} d\tilde{\xi} \\ &+ O(\log \frac{1}{\lambda}) \end{aligned}$$

So, (C.1), (C.2) and (C.3) give the asymptotics, when $\lambda \rightarrow 0$:

$$V(a; \lambda) = \lambda^{n/\sigma} (\alpha \log \frac{1}{\lambda} + \beta) + O(\log \frac{1}{\lambda})$$

with α given by (C.3) and β given by both (C.2) and (C.3).

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