Numerical approximation of the spectra of non-compact operators arising in buckling problems

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Abstract

The finite element method approximates the spectrum of an operator S by computing the spectra of a sequence of operators S_N defined in terms of the finite element spaces. For the case that S is compact, convergence of the approximate spectra follows from the convergence of S_N to S in the operator norm. We consider the case that S is non-compact, in which case such operator norm convergence cannot take place, and the approximations may be polluted by spurious eigenvalues. Pollution-free convergence of the eigenvalues can, however, be guaranteed outside the *essential numerical range* of S, which is related to the essential spectrum of S. We present results for estimating this essential numerical range and apply them to an algorithm for the buckling of three-dimensional bodies (that gives rise to a non-compact S). Our results show, for instance, that for the example of a circular disc, the algorithm will be free of spurious eigenvalues provided the body is thin enough. The case that singularities in the stresses can lead to non-physical spectral values being approximated is also investigated.

Key Words: buckling, eigenvalues, essential spectrum, thin domain, plate, shell, finite element method, p version, hp version

AMS(MOS) subject classifications (2000 revision): 35P15, 65N25, 65N30

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1 Introduction

The finite element approximation of the solution of eigenvalue problems has a long history. We refer, for example, to the monograph [2] and the references therein. The approximation theory is generally developed in terms of the spectrum of an operator $S : V \to V$ (V being an appropriate Sobolev space). The eigenvalues found by the FEM form the spectrum of another operator $S_N : V_N \to V_N$ (V_N being the finite element subspace of V). When S is compact, the operators S_N converge to S in the operator norm, and one can derive various optimal convergence results for the approximate eigenvalues and eigenvectors.

An example where eigenvalue problems arise is the determination of the *limit of elastic stability* of a body, or more informally, the point at which the body *buckles*. The mathematical formulation of this problem involves geometric non-linearities. One way to treat it is by incremental/continuation methods (see e.g. [20] and the references therein), where the required limit is determined as an inflection point in the load/displacement curve. An alternative strategy is to linearize the problem and determine the critical multiple of the applied load at which the equations fail to have a unique solution. This computed load can then be used as the starting point of a full non-linear analysis if needed. This second approach gives rise to an eigenvalue problem.

Buckling analysis is generally performed for structures that are *thin* in one dimension (such as rods, plates, shells, etc.). Traditionally, Kirchhoff-type assumptions are imposed on the displacements, to give a dimensionally-reduced model. This leads to the critical loads being formulated as the eigenvalues λ of a generalized eigenvalue problem of the form

$$T_\lambda x = (A - \lambda B)x = 0, \tag{1.1}$$

where typically A is a fourth-order differential operator and B is a second-order one. As a result, $\mu = \lambda^{-1}$ may be expressed as the eigenvalues of a compact operator S, and the theory from [2] is applicable.

Here, we consider the mathematical analysis of a method developed by SZABO and KIRALYFALVI in [23], and implemented in the hp commercial code STRESS CHECK. The underlying model is essentially one derived classically by TREFFTZ [24]. This formulation does not use dimensional reduction, but rather, works with the full three-dimensional domain. The advantage is that topological details such as stiffeners, as well as more general loads and boundary conditions (which may be inconsistent with Kirchhoff-type assumptions), may now be modeled as well. The disadvantage is that the operators A and B in (1.1) are now *both* of second order, so that the underlying operator S is *non-compact*.

The non-compactness of S can cause serious complications. First, the essential

spectrum of S no longer has to be a subset of $\{0\}$, as it must for compact S. This means that the spectrum $\sigma(S)$ of S may now contain, for example, eigenvalues of infinite multiplicity, accumulation points, a continuous spectrum, etc. Also, convergence results may no longer be guaranteed. Most serious of all, there might be spurious eigenvalues present in the approximation, i.e.

$$\mu_N \in \sigma(S_N), \quad \mu_N \to \mu_\infty \notin \sigma(S). \tag{1.2}$$

Spurious eigenvalues are known to occur in many cases when S is non-compact, for instance in problems in waveguides, magnetohydrodynamics, electromagnetics, etc. (see e.g. [12, 9, 14, 17]). A classical example is the so-called Cosserat problem (see [15] and the references therein), where

$$A = \Delta, \ B = \operatorname{grad}\operatorname{div}. \tag{1.3}$$

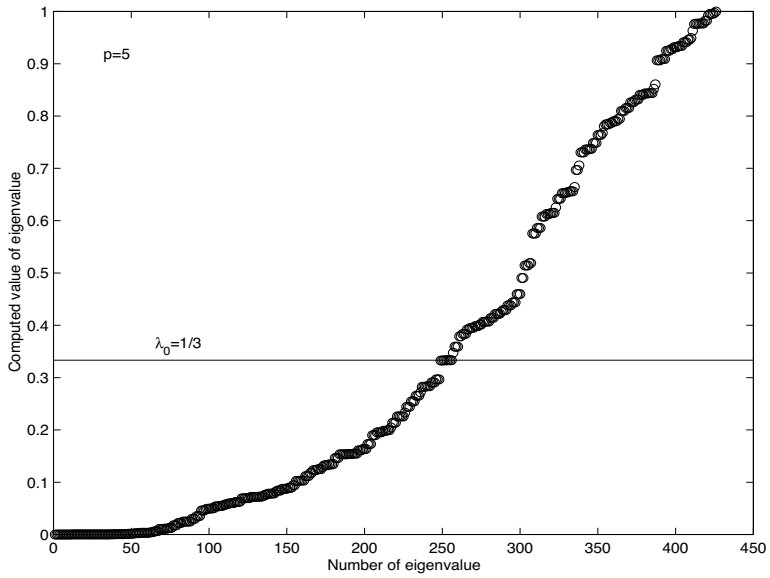

Figure 1: Approximate Cosserat e-values for p=5

For the unit ball with homogeneous Dirichlet boundary conditions, it is known [15] that the exact eigenvalues for the Cosserat problem are (m=multiplicity)

$$\lambda = 0, 1 \ \ (m = \infty) \quad \text{and} \quad \lambda = n/(2n+1) \ \ (m = n, \ n = 1, 2, \ldots). \tag{1.4}$$

Figures 1 and 2 show the results of a p version eigenvalue computation from [22] for this problem, using the code STRESS CHECK ($p = 5$ and 6 were used, with the sphere divided into 7 hexahedra — the x-axis just represents the numbering of the eigenvalues — first, second, etc.). It is seen that the eigenvalues are completely 'smeared,' with the results for $p = 5, 6$ suggesting that for *any* $\lambda_s \in [0, 1]$, there exists a sequence of

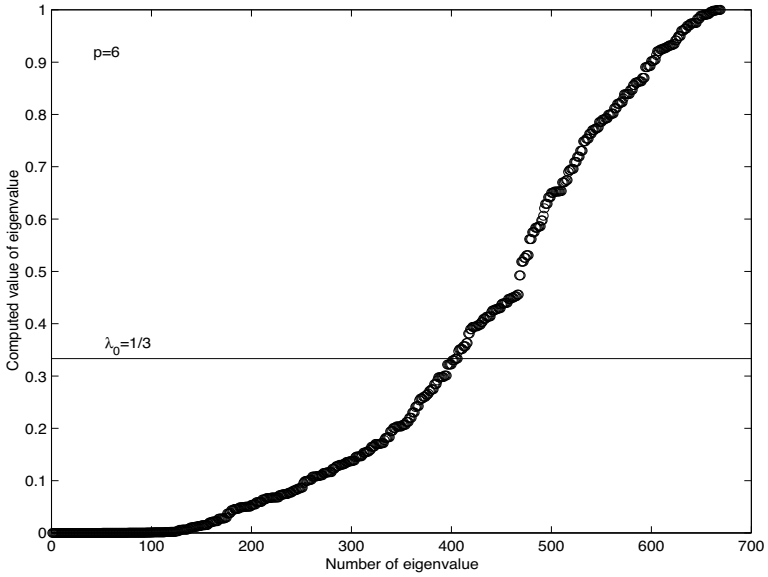

Figure 2: Approximate Cosserat e-values for p=6

finite element approximations λ_N converging to λ_s. Hence, it is impossible to estimate say the lowest positive eigenvalue $\lambda_0 = \frac{1}{3}$. This *spectral pollution* is caused by the eigenvalues of infinite multiplicity $\lambda = 0, 1$ — essentially, the spurious eigenvalues sequences are created by approximations trying to converge to these limits.

For many problems, spurious eigenvalues may be removed by modifying the underlying finite element method, so that the approximating subspaces satisfy the property in Remark 3.4 ahead. (Essentially, the operator S_N is designed to converge to S not in the operator norm, but a weaker one, which is sufficient for various desirable convergence properties.) Such conditions were first introduced in [9], and recently developed in the mixed method context in [4, 3]. (See also [2].) Implicit in this type of approach, however, is usually some underlying compactness in S which can be exploited to satisfy the necessary conditions. (For instance, in the examples considered in [9, 4], A has higher order derivatives than B for all except one unknown.) When this method is successful, one obtains pollution-free convergence to essentially the *entire* spectrum of A.

For the buckling formulation (as for the Cosserat problem), however, the operator S is observed to have *no* underlying compactness, and trying to design spaces that fit the criterion from [9] does not seem promising (see Remark 3.4). We therefore can only expect to approximate *part* of the spectrum without pollution. In Section 3.3, we present a key result on this by DESCLOUX [8] (see also [16, 18, 19]), which characterizes the portion of the spectrum (the complement of the so-called 'essential numerical range' of S) that can be approximated without pollution. In Section 4, we discuss two methods to estimate the essential numerical range. The first is based on characterizing the set of λ for which the operator T_λ in (1.1) is elliptic, while the second is based on the weak form

of the problem.

For arbitrary operators of equal order A, B, the criteria we develop still may not give us a region of the spectrum of S where pollution-free convergence is guaranteed. (As we shall see, the Cosserat problem is one such example.) However, for the case of buckling, we use the fact that such problems are of interest only when the domain is *thin* in one direction. In Section 5, we show for a sequence of domains of thickness d, satisfying a uniform Korn-type inequality as $d \to 0$, that the essential numerical range can be bounded independently of the thickness. Combining this with a result from [7], which shows that the eigenvalues of interest behave like $O(d^2)$ for the model problem of a thin plate then leads to our desired result of pollution-free eigenvalue approximation for such problems. Section 5.2 contains numerical examples illustrating our results for the case of a smooth domain.

Finally, in Section 6, we consider a case where the operator B is unbounded (which can occur, e.g. if the domain has corners or edges). We analyze some computational experiments to explain why one can still obtain buckling values of interest, even though the problem is not well-posed.

2 The linear model for buckling

Let $\Omega \subset \mathbb{R}^3$ denote the reference configuration of an elastic body, with the boundary $\partial\Omega$ being piecewise smooth. Suppose $\partial\Omega_T$ and $\partial\Omega_D$ are, respectively, the (disjoint) portions of $\partial\Omega$ where tractions and homogeneous displacements are to be specified. (We could also specify inhomogeneous displacements or spring (Newton) conditions on a portion of $\partial\Omega$ — see [23].)

Let us define for any functions $u = \{u_i\}, v = \{v_i\}$ on Ω (indices range from 1 to 3, repeated indices indicate summation), the bilinear form

$$a(u, v) = \int_\Omega C_{ijkl}\, u_{i,j}\, v_{k,l}\, dx, \tag{2.1}$$

where $C = \{C_{ijkl}\}$ is the tensor of elastic constants of the material and $u_{i,j} = \partial_j u_i$. This tensor is symmetric with respect to indices i, j and k, l, and moreover satisfies

$$C_{ijkl}\, \alpha_{ij}\, \alpha_{kl} \geq 0$$

for any matrix $\{\alpha_{ij}\}$. We define the admissible energy space by

$$V = \{u \in H^1(\Omega)^3, \quad u = 0 \text{ on } \partial\Omega_D\},$$

(we use standard notation for Sobolev spaces $H^k(\Omega)$) endowed with the norm

$$\|u\|_V := \Big(\sum_{i,j=1}^{3} \|\partial_j u_i\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}$$

(as $\partial\Omega_D \neq \emptyset$, $\|\cdot\|_V$ is really a norm on V). Then $a(\cdot,\cdot)$ is bounded in this norm, and by Korn inequality, a is strongly coercive.

Remark 2.1 In [23], the case $\partial\Omega_D = \emptyset$ is also treated, essentially by adding a (small) perturbation term to $a(u,v)$ to make it coercive. We could assume this modification as well when $\partial\Omega_D = \emptyset$ — see [23] for more details.

Suppose now that $\sigma^0 = \{\sigma^0_{ij}\}$ is a pre-existing stress state in the body. (For instance, σ^0 may be a *residual* stress, created in the body while the material was cooling after manufacture, or it could result e.g. from a thermal or other loading.) The stress σ^0 is already present in the reference configuration, it satisfies the equations of equilibrium (which would include any possible pre-existing loads on the body) and it is assumed to be independent of any subsequent displacements that the reference configuration may undergo.

If the reference configuration is now perturbed by a small change $\mathcal{F} \in V'$, the dual of V (which could be a change e.g, in body force, applied traction, thermal loading, etc), then the work done by σ^0 due to the product terms of the Green-Lagrange strain tensor cannot be neglected. The corresponding displacement u may be expressed as the solution of the following problem [23]. Find $u \in V$, satisfying, for all $v \in V$,

$$\int_\Omega C_{ijkl}\, u_{i,j}\, v_{k,l}\, dx + \int_\Omega \sigma^0_{ij}\, u_{m,i}\, v_{m,j}\, dx = \langle \mathcal{F}, v\rangle, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ is the duality between V' and V. The second ('geometric') term is the above-mentioned work done by σ^0. Writing

$$\sigma^0 = \lambda\sigma^* \tag{2.3}$$

for λ scalar, we see that the second term can be expressed as $-\lambda b(u,v)$ where

$$b(u,v) = -\int_\Omega \sigma^*_{ij}\, u_{m,i}\, v_{m,j}\, dx.$$

(Here, σ^* is called the 'pre-buckling' stress.)

Unless otherwise stated, σ^0 will be bounded on Ω. Then it is clear that $b(\cdot,\cdot)$ is a bounded bilinear form on V:

$$|b(u,v)| \leq 9M_*\|u\|_V\|v\|_V, \tag{2.4}$$

where

$$M_* = \max_{x,i,j} |\sigma^*_{ij}(x)|. \tag{2.5}$$

The case where $\partial\Omega$ is only piecewise smooth (or where the type of boundary condition changes) may result in a σ^0 which has unbounded singularities (such as at corners and

edges), leading to M_* in (2.5) being infinite. We discuss this case in Section 6, but note that since σ^* is pre-determined, it could be chosen to be smooth even when $\partial\Omega$ is not smooth.

We now write (2.2) as problem (P_λ) below.

(P_λ) Find $u \in V$ satisfying, for all $v \in V$,

$$a(u,v) - \lambda b(u,v) = \langle \mathcal{F}, v \rangle. \tag{2.6}$$

We will say that (P_λ) is *stably solvable* if (P_λ) has a unique solution for every $\mathcal{F} \in V'$, and there exists a constant C, independent of $\mathcal{F}$, such that

$$\|u\|_V \leq C \|\mathcal{F}\|_{V'}. \tag{2.7}$$

For instance, (P_0) is obviously stably solvable. Our goal will be to find the smallest positive λ (or the infimum of such λ's) for which (P_λ) is *not* stably solvable. This λ is called the *limit of elastic stability*. Physically, it represents the smallest multiple of the pre-buckling stress σ^* for which a small perturbation in external conditions on the body may cause it to buckle.

More generally, we consider the following question.

(Q) Find $\lambda \in \mathbb{C}$ for which (P_λ) is *not* stably solvable.

3 Some abstract results

Let V be a (complex) separable Hilbert space, with the compact imbedding $V \subset H$. Let $(\cdot,\cdot)$ be the inner product in H. Let the dual of V be V', with the duality pairing $\langle \cdot, \cdot \rangle$ being an extension of $(\cdot,\cdot)$. Let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be Hermitian, bounded, sesquilinear forms on $V \times V$ with $a(\cdot,\cdot)$ strongly coercive. We may then take a as Hilbertian product on V and define problem (Q) as above. In this section, we reduce (Q) to a spectral value problem, describe its spectrum, and consider its numerical approximation.

3.1 The spectral value problem

We define the operator $S : V \to V$ by: For any $w \in V, Sw \in V$ is the unique solution of

$$a(Sw, v) = b(w, v) \ \ \forall v \in V. \tag{3.1}$$

Lemma 3.1 *S is a self-adjoint operator on V with inner product a.*

Proof: Using the fact that $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are Hermitian, we have

$$a(Su, v) = b(u, v) = \overline{b(v, u)} = \overline{a(Sv, u)} = a(u, Sv). \tag{3.2}$$

The result follows. □

To see better what the operator S is, let us define operators $A : V \to V', B : V \to V'$ by

$$a(u, v) = \langle Au, v\rangle, \quad b(u, v) = \langle Bu, v\rangle. \tag{3.3}$$

Then A, A^{-1} are homeomorphisms between V and V', and (3.1) shows that

$$S = A^{-1}B. \tag{3.4}$$

Note that since A^{-1}, B are *bounded*, so is $S : V \to V$.

For any $\mu \in \mathbb{C}$, we now define the operator

$$S_\mu = \mu I - S. \tag{3.5}$$

Then the *resolvent* $\rho(S)$ of S [25] is the set of all $\mu \in \mathbb{C}$ for which S_μ is an *isomorphism* from V onto V. The complement of $\rho(S)$ is called the *spectrum* of S, denoted by $\sigma(S)$. We have the following theorem.

Theorem 3.2 (P_λ) *is not stably solvable for* $\lambda \in \mathbb{C}$ *if and only if* $\lambda^{-1} \in \sigma(S)$.

Proof: We write (2.6) as the following equation in V':

$$(A - \lambda B)u = \mathcal{F}. \tag{3.6}$$

Since $\lambda \neq 0$ (P_0 is always solvable), we may multiply equation (3.6) through by $\lambda^{-1}A^{-1}$ to obtain, with $\mu = \lambda^{-1}$,

$$(\mu I - S)u = \mu A^{-1}\mathcal{F}. \tag{3.7}$$

Since A^{-1} is a homeomorphism, the result follows easily by (3.6)-(3.7). □

Hence our problem reduces to finding the spectrum of S.

Remark 3.1 Note that $\mu = 0$ could be a spectral value of S, which does not correspond to any finite value of λ for which (P_λ) is not solvable.

3.2 Properties of the spectrum

We define the following components of the spectrum as in [13]:

(1) *Discrete spectrum*

$$\sigma_{\mathsf{ds}}(S) = \{\mu \in \mathbb{C},\ \ker S_\mu \neq \{0\} \text{ and } S_\mu \text{ is a Fredholm operator from } V \text{ into } V\}$$

(2) *Essential spectrum*

$$\sigma_{\mathsf{es}}(S) = \{\mu \in \mathbb{C},\ S_\mu \text{ is not a Fredholm operator from } V \text{ into } V\}$$

Then the self-adjointness of S immediately gives the following theorem [25].

Theorem 3.3 $\sigma(S) \subset \mathbb{R}$ *and* $\sigma(S) = \sigma_{\mathsf{es}}(S) \cup \sigma_{\mathsf{ds}}(S)$.

Points $\mu \in \sigma_{\mathsf{ds}}(S)$ are called *eigenvalues*, and for these, we may find an eigenvector $0 \neq u \in V$ for which $S_\mu u = 0$, or equivalently, for $\mu \neq 0$,

$$a(u, v) = \mu^{-1} b(u, v) \ \ \forall v \in V. \tag{3.8}$$

Also, $\mu = 0$ is an eigenvalue if and only if there is an eigenvector $0 \neq u \in V$ for which

$$b(u, v) = 0 \ \ \forall v \in V. \tag{3.9}$$

We also define the *Browder spectrum* $\sigma_{\mathsf{B}}(S)$ as the set of all $\mu \in \sigma(S)$ for which μ is not an isolated eigenvalue of finite (algebraic) multiplicity. An important property enjoyed by self-adjoint operators like S is the following (Page 518 of [13]):

$$\sigma_{\mathsf{es}}(S) = \sigma_{\mathsf{B}}(S).$$

For the case that S is, in addition, compact, it is well-known that $\sigma(S)$ consists only of eigenvalues of finite multiplicity with 0 being the only accumulation point (i.e. $\sigma_{\mathsf{es}}(S) \subset \{0\}$). However, the definition of $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ from Section 2 does *not* lead to a compact S.

Remark 3.2 Since here the spectrum is real (as will be any eigenvectors), we can reduce everything to the real case. Therefore, we assume in the sequel that λ, V are real.

3.3 Finite element approximation

Let $\{V_N\}$ be a family of finite-dimensional subspaces of V, parameterized by N, which will be identified with the dimension. We assume the following approximation property.

$$\forall u \in V, \exists u_N \in V_N \text{ such that } \lim_{N \to \infty} \|u - u_N\|_V = 0. \tag{3.10}$$

We now define an operator $S_N : V_N \to V_N$ by: For any $w \in V_N$, $S_N w \in V_N$ is the unique solution of

$$a(S_N w, v) = b(w, v) \ \ \forall v \in V_N. \tag{3.11}$$

Then S_N is the Galerkin approximation to S. If we define $\Pi_N : V \to V_N$ by

$$a(\Pi_N w, v) = a(w, v) \ \ \forall v \in V_N, \tag{3.12}$$

then it is seen that $S_N = \Pi_N S|_{V_N}$.

Since V_N is finite-dimensional, $\sigma(S_N)$ will consist only of eigenvalues μ_N. Non-zero μ_N can be found by solving the finite-dimensional (generalized) eigenvalue problem: Find $(\mu_N, u_N) = (\lambda_N^{-1}, u_N) \in \mathbb{R} \times V_N \setminus \{0\}$ satisfying, for all $v \in V_N$,

$$a(u_N, v) = \lambda_N b(u_N, v). \tag{3.13}$$

Also, 0 will be an eigenvalue of S_N if and only if $\exists\, u_N \in V_N \setminus \{0\}$ for which

$$b(u_N, v) = 0 \ \ \forall v \in V_N. \tag{3.14}$$

For the case that S is *compact*, it is well-known (see e.g. [2]) that the (non-zero) eigenvalues of S_N converge to those of S at optimal rates (as do the eigenvectors), and there are no spurious eigenvalues. However, since S is not compact in our case, spurious eigenvalues satisfying (1.2) may exist, like in the Cosserat problem. As mentioned in the introduction, it is sometimes possible to design a finite element method which satisfies certain conditions [9] for the pollution-free approximation of *all* non-zero eigenvalues. This approach is discussed in Remark 3.4 at the end of this section, but it is not promising in our case. We therefore present an alternative theorem which characterizes those eigenvalues that may be approximated without pollution when the standard finite element method (3.13) is used, under only the assumption (3.10).

Let us define

$$W_{\mathsf{es}} = [\min \sigma_{\mathsf{es}}(S), \max \sigma_{\mathsf{es}}(S)], \ \ \Lambda = \mathbb{C} - W_{\mathsf{es}}. \tag{3.15}$$

W_{es} is called the *essential numerical range* of S. The following result is established in [8, 18].

Theorem 3.4 *Let Δ be a compact subset of Λ.*
(A) *If $\Delta \cap \sigma(S) = \emptyset$, then there exists an integer $N_0 > 0$ and a constant $C > 0$ such that*

$$\begin{aligned} \Delta \cap \sigma(S_N) &= \emptyset \qquad \text{for all } N \geq N_0 \\ \|(\mu I - S_N)v\|_V &\geq C\|v\|_V \quad \forall N \geq N_0,\ \forall v \in V_N,\ \forall \mu \in \Delta. \end{aligned}$$

(B) *If Δ is a neighborhood of an isolated $\mu \in \sigma_{\mathsf{ds}}(S)$ with finite algebraic multiplicity m and $\Delta \cap \sigma(S) = \{\mu\}$, then there exists an integer $N_0 > 0$ such that for $N \geq N_0$, $\Delta \cap \sigma(S_N)$ consists of exactly m eigenvalues $\mu_N^1, \mu_N^2, \ldots \mu_N^m$ of S_N (counted according to algebraic multiplicity). Moreover, $\lim_{N\to\infty} \mu_N^i = \mu$, for $i = 1, 2, \ldots m$.*

We may also obtain optimal error estimates for $|\mu - \frac{1}{m}\Sigma_{i=1}^m \mu_N^i|$. Note in particular that (A) ensures there are no spurious eigenvalues in the approximation of the spectrum that lies in Λ. On the other hand, as shown in [8], one can construct finite element spaces such that there is a sequence of approximate eigenvalues μ_N converging to *each* $\mu_\infty \in W_{\mathsf{es}}$. This is what appears to be happening for the Cosserat problem in Figures 1 and 2: By (1.4), we have

$$W_{\mathsf{es}} = [0, 1]$$

and *every* number in $[0, 1]$ may be a limit of a sequence of approximations.

Remark 3.3 The characterization (3.15) can be refined to yield a larger Λ, see [18, 19].

Remark 3.4 Let us define the following norm for an operator on V_N (or V).

$$\|S\|_N = \sup_{v \in V_N,\, \|v\|_V = 1} \|Sv\|_V. \tag{3.16}$$

It is shown in [9] that the following condition:

$$\lim_{N \to \infty} \|S - S_N\|_N = 0 \tag{3.17}$$

combined with (3.10) is sufficient for Theorem 3.4 to hold over $\Lambda = \mathbb{C}$. (In fact, (3.17) is *necessary* as well for the property θ in [9] to hold.)

As stated in the introduction, (3.17), which is weaker than convergence in the operator norm, can be made to hold for certain problems by designing the finite element methods appropriately (see [9, 17]). A similar strategy for mixed methods may be found in [4, 3]. Unlike the problems discussed in the above references, however, the buckling problem and the Cosserat problem appear to have no inherent compactness to make such an approach successful. For instance, it is easily verified that for the Cosserat problem, condition (3.17) reduces to

$$\lim_{N \to \infty} \sup_{v \in V_N,\, \|v\|_V = 1} \inf_{w \in V_N} \|Sv - w\|_V = 0, \tag{3.18}$$

with $S = A^{-1}B$. It is not apparent how to design finite element spaces for which (3.18) will hold when A, B are as in (1.3). The situation for the general 3-d buckling problem is analogous.

4 Estimation of the essential numerical range

Theorem 3.4 shows that in order to characterize the region Λ that is free of spectral pollution, we must estimate the essential numerical range given in (3.15), i.e. we must characterize the essential spectrum $\sigma_{\mathsf{es}}(S)$. There holds

Theorem 4.1 *For $\lambda \in \mathbb{R}, \mu = 1/\lambda \in \sigma_{\mathsf{es}}(S)$ if and only if $T_\lambda = A - \lambda B$ is not Fredholm from V into V'.*

Proof: Noting that A is an isomorphism from V into V', we see by multiplying (3.5) by $\mu^{-1}A$ that S_μ is Fredholm if and only if T_λ is Fredholm. The result follows easily by the definition of the essential spectrum. □

We discuss two methods to characterize (or provide an estimate for) the set of λ's such that T_λ is Fredholm.

4.1 Estimation using ellipticity of operators

The first method uses the fact that in applications, A, B can be identified with partial differential operators. For example, for the Cosserat problem, we obtain the second-order operators given by (1.3). We may now consider the ellipticity or the non-ellipticity of the operator T_λ in the sense of Agmon Douglis Nirenberg [1]. Let us define

$$\omega_1 = \{\lambda, \;\; T_\lambda \text{ is not elliptic everywhere on } \overline{\Omega}\}$$

The choice of *essential boundary conditions* in V (i.e. the part $\partial\Omega_D$ where u satisfies Dirichlet boundary conditions) determines boundary conditions for the operators T_λ. We also have to check whether these boundary conditions are complementing or not for T_λ. We thus define

$$\omega_2 = \{\lambda, \;\; \text{boundary conditions are not complementing everywhere on } \partial\Omega \text{ for } T_\lambda\}.$$

Then, in the case when

$$\partial\Omega \text{ is smooth} \quad \text{and} \quad \partial\Omega_D \text{ is a connected component of } \partial\Omega\,, \tag{4.1}$$

T_λ is Fredholm (as an operator from $H^2(\Omega) \cap V$ into $L_2(\Omega)$) if and only if it is elliptic on $\overline{\Omega}$ with complementing boundary conditions on $\partial\Omega$ (see e.g. Lemma 2 of [21]) i.e. if and only if $\lambda \notin \omega_1 \cup \omega_2$.

The sets ω_1 and ω_2 can be found by checking algebraic conditions (for instance ω_1 by setting the determinant of the symbol of the highest-order terms of T_λ to be zero). See [15], where this procedure has been carried out for the Cosserat problem, yielding $\omega_1 = \{0, 1\}$ and $\omega_2 = \{1/2\}$ in the Dirichlet case (i.e. when $\partial\Omega_D$ coincides with the whole boundary $\partial\Omega$).

In order to use Theorem 4.1, however, we need to consider T_λ as an operator from V into its dual (rather than from $H^2(\Omega) \cap V$ into $L_2(\Omega)$). Theorem 4 of [21] says, essentially, that this does not change the values of λ for which T_λ is Fredholm. However, there are technicalities involved, which may be difficult to resolve, since spaces V and V' have to be carefully defined to incorporate appropriate classes of boundary data (see [21]).

Although we are not aware of a rigorous proof, we expect that (once again under condition (4.1)) we have

$$\sigma_{\mathsf{es}}(S) = \omega_1 \cup \omega_2. \tag{4.2}$$

See [10], where some of the technicalities are resolved for the case $T_\lambda = A - \lambda I$.

For the Cosserat problem with zero Dirichlet conditions, such a characterization has been used in [15] to give $\sigma_{\mathsf{es}}(S) = \{0, 1/2, 1\}$.

However, applying this to the linear buckling problem brings more difficulties, since the conditions (4.1) will not be satisfied in general. For instance, for the model case of a thin circular plate, with the free boundary on the top and bottom surfaces and the clamped condition on the lateral surface, we would have to consider a new ellipticity condition along the edges of the plate, where the boundary is not smooth and the boundary conditions change in nature, giving rise to a third set ω_3, which has to be added to the first two.

Moreover, without knowing the explicit form of σ^* which occurs in the definition of B, we cannot expect to find $\sigma_{\mathsf{es}}(S)$ exactly. We therefore present a second approach below, which is useful for estimating $\sigma_{\mathsf{es}}(S)$, rather than characterizing it exactly.

4.2 Estimation using uniform coercivity

Here, we use the variational form of the problem rather than the operators A, B. We first note the following key result.

Lemma 4.2 *Let $\lambda \in \mathbb{R}\backslash\{0\}$ be such that $\forall u \in V$,*

$$a(u,u) - \lambda b(u,u) \geq C_1\|u\|_V^2 - C_2\|u\|_H^2$$

with $C_1 > 0, C_2$ constants independent of u. Then $S_{1/\lambda} = \lambda^{-1}I - S$ is Fredholm from $V \to V$.

Proof: The form

$$\{u,v\} \to a(u,v) - \lambda b(u,v) + C_2(u,v)$$

is strongly coercive on V. Therefore, the operator

$$A - \lambda B + C_2 I : V \to V'$$

is invertible. Hence, composing by $\lambda^{-1}A^{-1}$ to the left,

$$(\lambda^{-1}I - S) + C_2\lambda^{-1}A^{-1} : V \to V$$

is invertible. Now A^{-1} as an operator from V into V is compact. Therefore, $S_{1/\lambda}$ is Fredholm. □

Lemma 4.2 leads to the following result.

Theorem 4.3 *Let α, β, γ be positive constants such that $\forall u \in V$,*

$$a(u,u) \geq \alpha\|u\|_V^2 - \gamma\|u\|_H^2, \tag{4.3}$$

$$|b(u,u)| \leq \beta\|u\|_V^2. \tag{4.4}$$

Then $\sigma_{\mathsf{es}}(S) \subset [-\frac{\beta}{\alpha}, \frac{\beta}{\alpha}]$. Therefore the essential numerical range $W_{\mathsf{es}} \subset [-\frac{\beta}{\alpha}, \frac{\beta}{\alpha}]$.

Proof: For any $\lambda \in \mathbb{R}$, we have, using (4.3) and (4.4),

$$\begin{aligned} a(u,u) - \lambda b(u,u) &\geq a(u,u) - |\lambda||b(u,u)| \\ &\geq a(u,u) - |\lambda|\beta\|u\|_V^2 \\ &\geq (\alpha - |\lambda|\beta)\|u\|_V^2 - \alpha\gamma\|u\|_H^2. \end{aligned}$$

Let $|\lambda| < \alpha/\beta$. We may then take

$$C_1 = \alpha - |\lambda|\beta, \;\; C_2 = \alpha\gamma$$

in Lemma 4.2 to conclude that $S_{1/\lambda}$ is Fredholm. Hence, $\lambda^{-1} \notin \sigma_{\mathsf{es}}(S)$ and the result follows. □

At first glance, when $a(\cdot,\cdot)$ is strongly coercive, it might seem natural to take $\gamma = 0$ in (4.3). However, in this case, the theorem does not give any useful information, since then the *entire* spectrum $\sigma(S)$ will be a subset of the indicated interval. The usefulness of the theorem arises when we are able to choose $\gamma > 0$ and thereby increase the value of α for which (4.3) will still hold. We illustrate this by an application to the linear buckling problem in the next section.

5 The buckling problem over thin domains

5.1 The theoretical framework

If we apply the results of Section 4.1 to our Cosserat problem example, we see by Theorem 3.4 that the region of spectral pollution is limited to $W_{\mathsf{es}} = [0,1]$. However, since by (1.4) the *entire* spectrum lies in W_{es}, we are unable to guarantee pollution-free convergence to any of the eigenvalues of interest, and this is what is illustrated in Figures 1 and 2.

Turning to the buckling problem over a general three-dimensional domain, we see that the structure of the equations is similar to the Cosserat problem. Depending on the domain (and the pre-buckling stress σ^*), we cannot eliminate the possibility that a similar situation (of the entire spectrum being polluted) may occur.

As stated in the introduction, however, physically interesting problems of buckling occur over domains that are *thin* in one dimension (such as plates, shells, etc.). Suppose we are given a sequence of such domains $\{\Omega_d\}$, parameterized by the 'thickness' variable $d \in (0,1]$. Then we assume $\{\Omega_d\}$ satisfies the following weak *uniform Korn inequality*.

Korn Inequality $\exists$ constants $K > 0$ and $r \geq 0$ such that the following inequality holds uniformly $\forall d \in (0,1], \forall u \in V$:

$$\|u\|_V^2 \leq K\big(a(u,u) + d^{-r}\|u\|_{L_2(\Omega_d)}^2\big). \tag{5.1}$$

The above inequality (5.1) has been established with $r = 2$ for thin plates of the form $\Omega_d = \omega \times (-\frac{d}{2}, \frac{d}{2})$ with $\omega \in \mathbb{R}^2$ in [6]. Similar results can be established for thin shells (with regular mid-surface).

The following result holds for domains satisfying (5.1).

Theorem 5.1 *Let* $\{\Omega_d\}$ *be such that the uniform Korn inequality* (5.1) *holds. Let the pre-buckling stress* σ^* *have the bound* M_* *as in* (2.5)*. Then* $\forall d \in (0, 1]$,

$$\sigma_{\mathsf{es}}(S) \subset [-9KM_*, 9KM_*].$$

Proof: We apply Theorem 4.3, noting that (5.1) implies (4.3) is satisfied with $\alpha = K^{-1}, \gamma = d^{-r}$, and that (2.4) gives (4.4) with $\beta = 9M_*$. □

Theorem 5.1 shows that for such domains, there will be a region $(-\infty, -9KM_*) \cup (9KM_*, \infty)$ independent of d, which is free of spectral pollution. (Note that without using (5.1) and Theorem 4.3 we could not conclude that the above region was independent of d.) Let us denote

$$\Lambda = (-\infty, -\mu^-_{\min}) \cup (\mu^+_{\min}, \infty) \supset (-\infty, -9KM_*) \cup (9KM_*, \infty) \tag{5.2}$$

to be the maximal such pollution-free region. Then if the eigenvalues that are of interest to us lie inside this domain, we will be able to recover them accurately. Note that we are interested in the *highest* values μ in the spectrum $\sigma(S)$, corresponding to the lowest values of λ. In terms of λ, we may take $\lambda^\pm_{\max} = 1/\mu^\pm_{\min}$ to write the above maximal region as

$$\widetilde{\Lambda} = (-\lambda^-_{\max}, \lambda^+_{\max}). \tag{5.3}$$

Fortunately, in most practical cases of interest, we can expect for the lowest several buckling eigenvalues λ_d corresponding to domain Ω_d,

$$\lambda_d \to 0 \quad \text{as} \quad d \to 0. \tag{5.4}$$

Physically, this just expresses the fact that as the domain gets thinner, it will take a smaller multiple of the pre-buckling stress σ^*_d to make it buckle. (We are assuming we are given an appropriate family of pre-buckling stresses $\{\sigma^*_d\}$, such that they converge as $d \to 0$ to an appropriate σ^*_0.)

In [7], taking advantage of the spectral analysis in thin plates investigated in [5], we prove (5.4) rigorously for the family of plates $\omega \times (-\frac{d}{2}, \frac{d}{2})$ mentioned above: We show that as long as the pre-buckling stress has a non-zero membrane component (which is the case of practical interest), the following estimate holds uniformly in d for the lowest eigenvalue $\lambda^{\min}_d$:

$$|\lambda^{\min}_d - d^2\lambda^{\min}_0| \le Cd^3. \tag{5.5}$$

Here $\lambda_0^{\min}$ is the lowest eigenvalue of a limiting problem which is compact: as $d \to 0$, the operator A in (1.1) essentially tends to the fourth-order Kirchhoff operator, while B remains second-order. Hence, the limiting S is compact, and the eigenvalues $\lambda_0 = 1/\mu_0$ are well-separated from 0. Estimate (5.5) will hold for not just the minimum, but a whole family of eigenvalues.

Remark 5.1 The existence of eigenvalues behaving as $d^2\lambda_0$ as $d \to 0$ is related to the possibility of exciting *bending modes* in the thin structure. For example, we do not expect such a behavior in the case of *clamped elliptic shells*.

We therefore see by (5.3) and (5.5) that for the plate problem, with σ^* not of pure-bending type, we will obtain no spectral pollution in the desired minimum buckling eigenvalues, once the thickness d is small enough. More generally, we have the following theorem.

Theorem 5.2 *Let $\{\Omega_d\}$ be a sequence of domains parameterized by a thickness parameter d, such that the uniform Korn's inequality* (5.1) *is satisfied. Then for any sequence of eigenvalues $\{\lambda_d\}$ satisfying* (5.4)*, the finite element method gives pollution-free approximations (in the sense of Theorem* 3.4*) for d small enough.*

5.2 Numerical results

We now present numerical experiments for a model problem that demonstrate that spectral pollution will occur only for rather large d, well above the range of practical interest.

We take $\{\Omega_d\}$ to be the family of isotropic circular plates of unit radius and uniform thickness d. Elastic constants are given by $E = 3 \times 10^4$ and $\nu = 0.3$.

Let $x_* = (x_1, x_2)$ be the variables in the mid-surface, with origin at the center of the circle, and let $x_3 \in (-\frac{d}{2}, \frac{d}{2})$ be the variable in the thickness direction. The corresponding components of the displacement u are $(u_1, u_2, u_3) = (u_*, u_3)$. By varying d, we can consider the difference between a thin plate and a cylindrical domain. The uniform Korn inequality (5.1) will hold for this family of domains, with $r = 2$.

The lateral boundary condition is soft simple support, i.e. the component u_3 is zero along the lateral cylindrical boundary. The two circular faces are left free.

We load the disc by a uniform inward radial load of unity along the lateral cylindrical boundary. This load is symmetric (membrane symmetry) with respect to the plane $x_3 = 0$ and any plane $x_1 \sin\theta = x_2 \cos\theta$.

Let Σ be the transverse symmetry operator defined for $u = (u_*, u_3)$ by:

$$u \xrightarrow{\Sigma} \Big((x_*, x_3) \mapsto \big(u_*(x_*, -x_3),\ -u_3(x_*, -x_3) \big) \Big).$$

Then $A\Sigma = \Sigma A$ and since the pre-buckling stress originates from a load u satisfying $\Sigma(u) = u$ (i.e. a membrane load), we also obtain that $B\Sigma = \Sigma B$. This is also valid for

the symmetries Σ_θ associated with the plane $x_1 \sin\theta = x_2 \cos\theta$. Therefore the buckling eigen-spaces are invariant under both Σ and $-\Sigma$ and similarly for any Σ_θ.

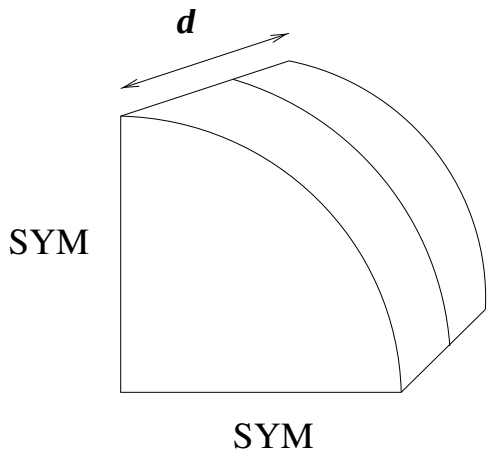

Figure 3: The quarter disc with thickness d

Using the symmetries of the domain, we take as computational domain the quarter of the plate as shown in Figure 3, imposing a sliding edge boundary condition on the plane lateral parts of the boundary $x_1 = 0$ and $x_2 = 0$ (i.e. the displacement in the normal direction to these planes is 0, so that symmetry is enforced in the solution on the full domain Ω_d across these planes). We then compute approximations for those eigenvalues whose eigenvectors satisfy these symmetries over the full domain (roughly a quarter of the total number). The lowest eigenvalue for this example turns out to have a symmetric eigenvector with respect to any plane $x_1 \sin\theta = x_2 \cos\theta$, i.e. it is a radial eigenvector.

The resulting stress is calculated with 8 elements using STRESS CHECK, with degree $p = 8$, for various values of d. (The quarter of the plate is sliced into two halves, each of which has 4 elements.) The calculated stress is used as the pre-buckling stress σ_d^* in each case, and the lowest 10 eigenvalues computed. Figure 4 shows the computed eigenvalues.

It is observed that there is a value, $\lambda^+_{\max} \sim 6600$, such that eigenvalues that satisfy $\lambda < \lambda^+_{\max}$ are computed accurately (as verified by comparisons with the results for lower p, which are not shown here). However, no values of $\lambda > \lambda^+_{\max}$ are returned — instead, these values simply seem to be 'absorbed' into the value of $\lambda^+_{\max}$. (Computing the lowest 50 eigenvalues also did not return anything larger than $\lambda^+_{\max}$.) Our interpretation of this phenomenon is that $\lambda^+_{\max} \in \sigma_{\mathsf{es}}(S)$ and represents the smallest value of λ for which the operator $A - \lambda B$ is no longer elliptic with complementing boundary conditions. The part of the spectrum that lies above $\lambda^+_{\max}$ is subject to pollution, and we are unable to recover it. For instance, for $d = 1.4$ and larger, the *entire* spectrum lies above $\lambda^+_{\max}$.

In Figure 5, we plot the computed eigenvalues as functions of d. The slopes clearly demonstrate the $O(d^2)$ behavior predicted by (5.5). This shows that when d is small, we

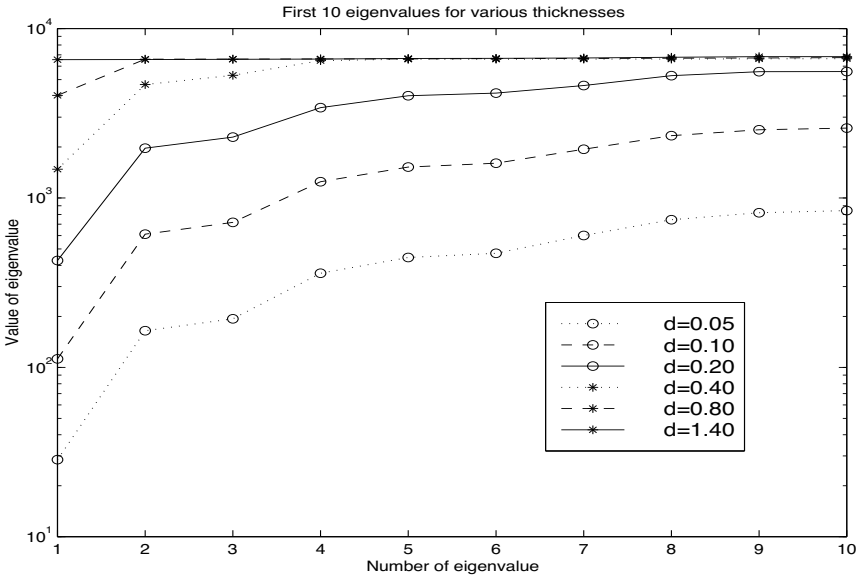

Figure 4: The first ten eigenvalues for different d

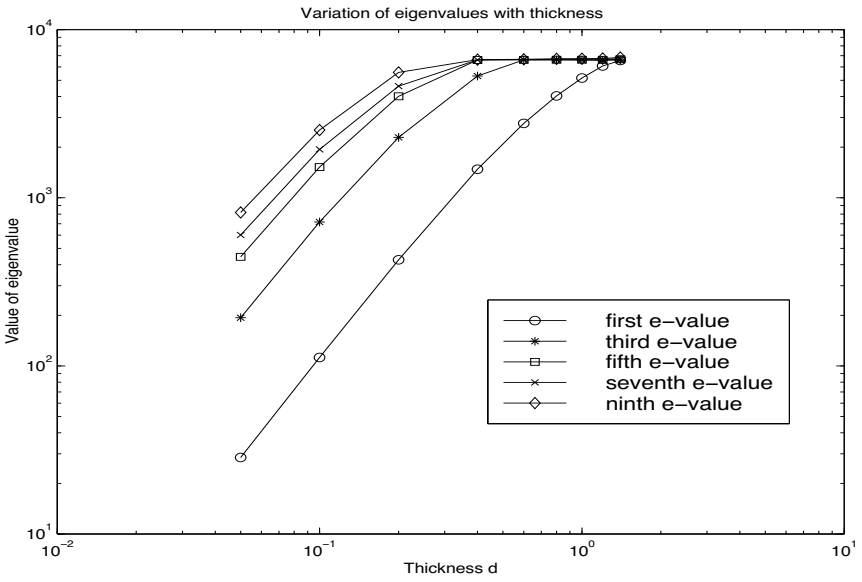

Figure 5: The $O(d^2)$ behavior of the physical eigenvalues

can expect to calculate several buckling modes, before the cut-off maximum of $\lambda^+_{\max} \sim 6600$ is reached. As d increases, these buckling multiples increase as well, and fewer of them can be calculated, until we reach a critical thickness (about $d = 1.4$ in this case), where all we can calculate is $\lambda^+_{\max}$, the point where ellipticity is lost.

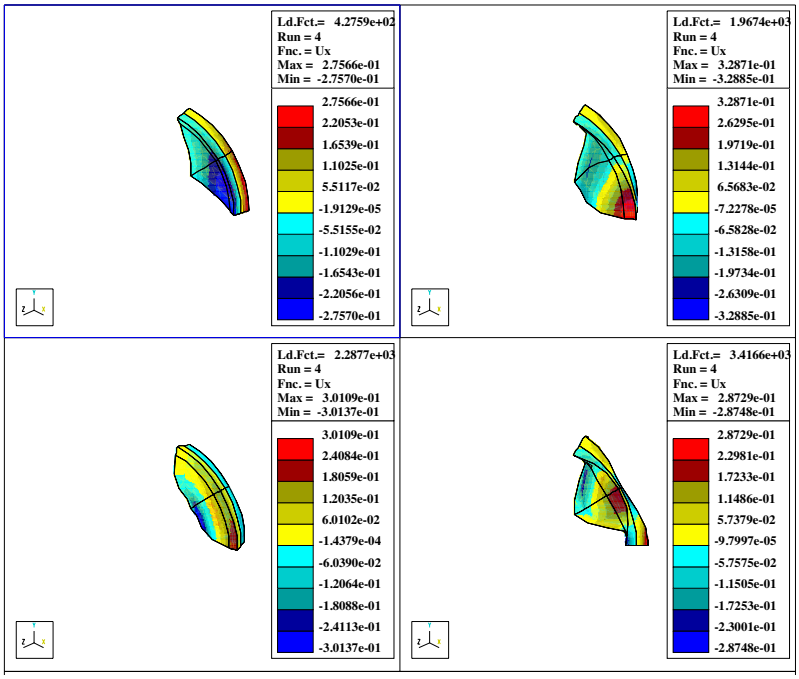

Figure 6: The first four eigenvectors, $d = 0.2$

The difference between 'physical' and 'non-physical' (like $\lambda^+_{\max}$) modes comes out quite clearly when we examine the corresponding eigenvectors (which represent the buckling deformations). Figures 6-9 show this difference, where the u_1 component of the first four buckling eigenvectors are plotted for $d = 0.2$, 0.4, 0.8 and 1.4, which have 4, 3, 1 and 0 physical eigenvalues respectively. There is a marked difference in the corresponding deformations. We note in particular that, as expected, the physical buckling modes have the bending symmetry (i.e. u_1 is *odd* with respect to the plane $x_3 = 0$) whereas no such anti-symmetry is visible for the non-physical modes.

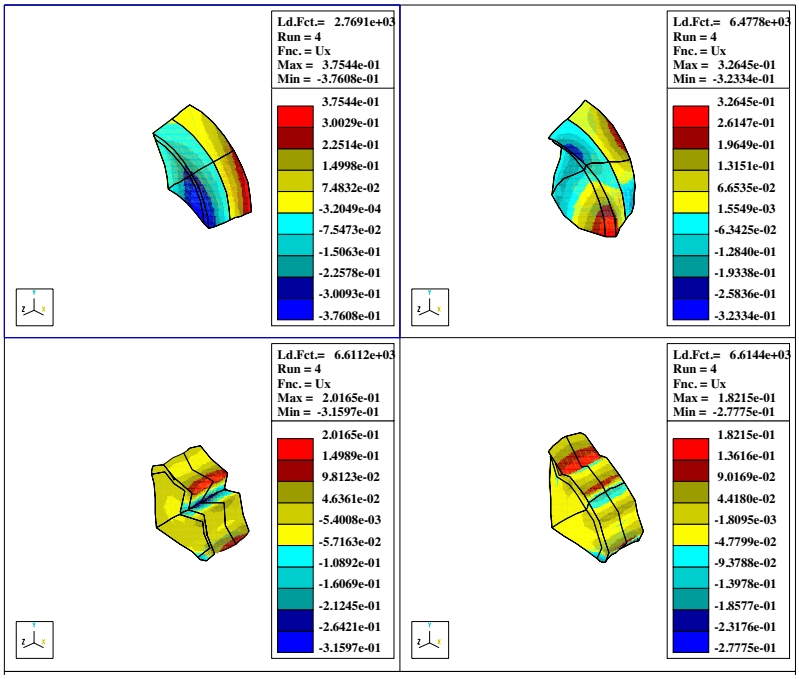

Figure 7: The first four eigenvectors, $d = 0.4$

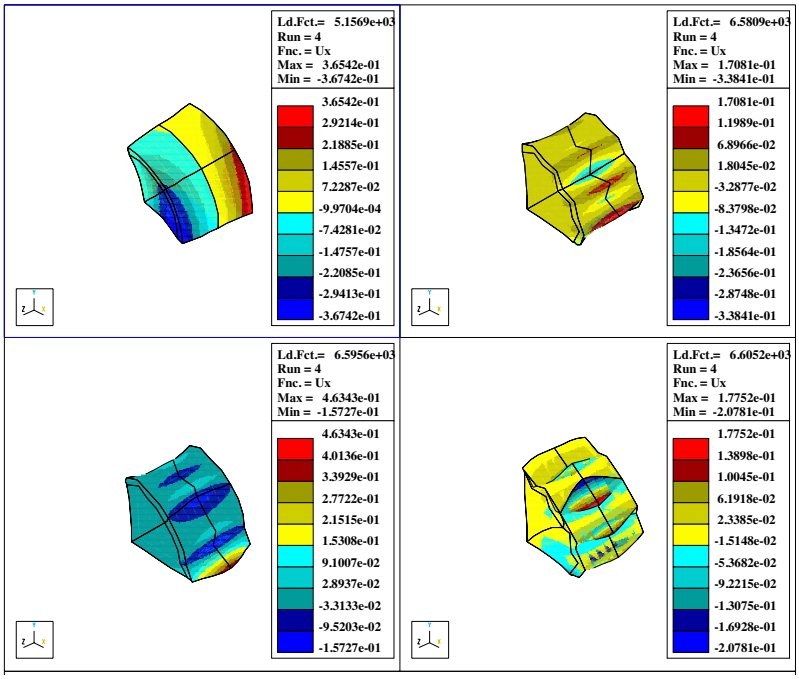

Figure 8: The first four eigenvectors, $d = 0.8$

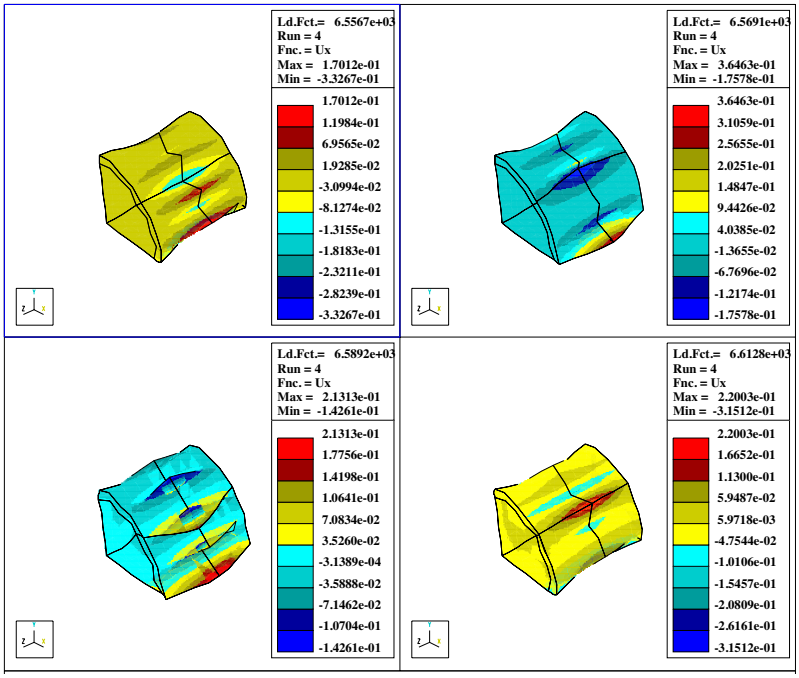

Figure 9: The first four eigenvectors, $d = 1.4$

6 The case of unbounded σ^*

So far, we have considered that σ^* is bounded. However, if the region Ω has corners or edges, then this condition could be violated. For instance, the algorithm in STRESS CHECK uses a representative σ^* that is first computed by considering the equations of equilibrium on the body under any external forces (or displacements) that are applied, and then inputs this pre-buckling stress into the eigenvalue calculation. Consequently, we may expect σ^* (if calculated with infinite accuracy) to be unbounded at edges and corners.

For unbounded σ^*, we see that M_* in (2.5) is infinite, so that the form $b(\cdot,\cdot)$ is unbounded, and the three-dimensional buckling problem is not well-formulated. However, the dimensionally reduced (Kirchhoff) case can still be well-posed. This is because now A will be fourth-order and B second-order, so that the energy space will be a subset of $H^2(\Omega)$ rather than $H^1(\Omega)$. Although $b(\cdot,\cdot)$ will not be bounded in the $H^1(\Omega)$ norm, it might still be bounded in the energy norm, since with proper boundary conditions a singularity that is not too strong can be absorbed into the $H^2(\Omega)$ norm. Hence the Kirchhoff limit gives a bounded, compact eigenvalue problem, and we can compute its lowest positive eigenvalue $\lambda_0^{\min}$. See [7], where the above is illustrated for a thin plate.

Although (5.5) may not hold when the domain is not smooth, we expect that for finite thicknesses, there will still be a sequence of eigenvalues $\{\lambda_d^{\min}\}$ that converge to $\lambda_0^{\min}$, as in the smooth case. The purpose of the engineering calculation is generally to find this $\lambda_d^{\min}$ for given $d > 0$. Of course, once $d > 0$, we are not in the Kirchhoff limit, and the problem becomes ill-posed due to loss of boundedness. Applying Theorem 5.1, we will be able to find $\lambda_d^{\min}$ provided (see (5.3))

$$\lambda_d^{\min} \in \widetilde{\Lambda}, \text{ i.e. } \lambda_d^{\min} < \lambda_{\max}^{+}. \tag{6.1}$$

However, as the stress becomes unbounded, i.e. $M_* \to \infty$, we expect $\lambda_{\max}^{+} \to 0$ which will preclude (6.1) from holding.

In actual practice, however, our computed stress may be large but is never unbounded, since we do not have infinite accuracy but use $V_N \subset V$ instead. We can therefore expect a sequence $\{\sigma^*(N)\}$ for which the maximum $M_*(N)$ in (2.5) tends to (but never attains) infinity as $N \to \infty$ (see (6.3) for an $L_2(\Omega)$ estimate). The corresponding $\lambda_{\max}^{+}(N)$ in (5.3) can be expected to degenerate to 0, as observed in the numerical example below. By (6.1), our computation of $\lambda_d^{\min}$ can then only succeed if

$$\lambda_d^{\min} < \lambda_{\max}^{+}(N). \tag{6.2}$$

Equation (6.2) suggests that in order to obtain pollution-free approximations to $\lambda_d^{\min}$, we should either have d small enough (which we have already seen in the previous section),

or, paradoxically, the level of discretization N not too large. This is observed in the numerical experiments that follow.

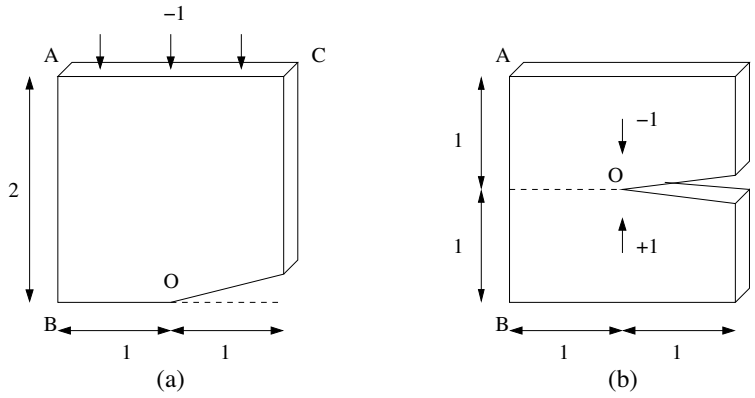

Figure 10: Cracked panels.
(a) Panel A ($E = 3 \times 10^7, \nu = 0$) (b) Panel B ($E = 3 \times 10^{10}, \nu = 0.3$)

We first consider Panel A (Figure 6(a)), which is the top right quarter of a rectangular panel with a symmetric center crack (the angle to the horizontal shown in the figure is 5 degrees). We use symmetry conditions at the base BO and the side AB. Soft simple support boundary conditions are applied on the top face AC and the panel is subjected to a uniform traction of -1 on this face (and symmetrically, on the bottom face). The strongest singularity will now lie at the vertex of the crack, which must be resolved to compute the representative pre-buckling stress σ^* accurately. We therefore use hp geometric refinement in the vicinity of this vertex (see [11]) with the number of geometric layers denoted by n and the polynomial degree in each element by p (so that $N \sim np^2$). By [11], we can expect the following exponential estimate for the approximation to the pre-buckling stress.

$$\|\sigma^* - \sigma^*(N)\|_{L_2(\Omega)} \leq Ce^{-\gamma \sqrt[5]{N}}. \tag{6.3}$$

After $\sigma^*(N)$ has been calculated, it is used in the buckling formulation to find the corresponding buckling eigenvalues. Figure 11 shows the computations of the lowest two eigenvalues for various d when p is fixed at 8 and the number of layers n is varied. We make the following observations:

1. As in the smooth case, the lowest buckling eigenvalue $\lambda_d^{\min} \to 0$ at the rate $O(d^2)$. The same is observed for the next eigenvalue.

2. The computed eigenvalues (say $\lambda_d^{\min}$) have already converged for $n = 1$ layer, so that they do not require much refinement at the vertex. This may be explained by buckling being a global phenomenon, which is not very sensitive on exactly recovering the singularity at the vertex.

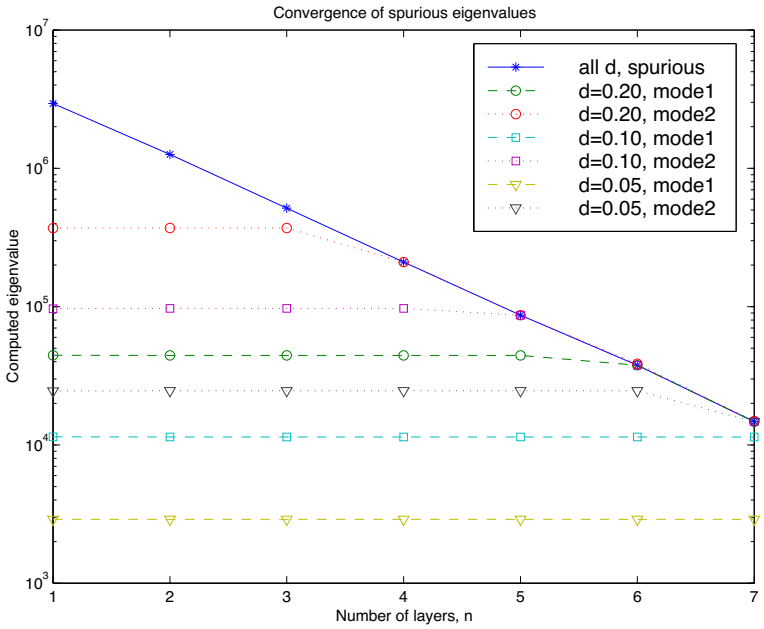

Figure 11: Computed eigenvalues for cracked panel

3. As in the smooth case, there is an 'absorbing' value $\lambda^+_{\max}$ such that no eigenvalue over $\lambda^+_{\max}$ can be recovered, and $\lambda^+_{\max}$ is independent of d.

4. Unlike the smooth case, $\lambda^+_{\max} = \lambda^+_{\max}(N)$ depends on the amount of discretization. In fact, we observe

$$|\lambda^+_{\max}(N)| \sim Ce^{-\gamma n}, \tag{6.4}$$

   i.e. $\lambda^+_{\max}(N)$ tends exponentially to 0 with the number of layers. This is consistent with (6.3) and our explanation above — the larger is the bound $M_*(N)$ (which increases exponentially to ∞), the smaller will be $\lambda^+_{\max}$ (which decreases exponentially to 0).

5. Figure 11 clearly illustrates the condition (6.2) that must hold for the successful computation of the required buckling eigenvalues. Due to the concentrated power of the hp version in resolving singularities, the danger is over- rather than under-refinement. Of course, no matter what method is used (e.g. h or p version), similar results (of $\lambda^+_{\max}$ being reached) will be observed once the refinement has proceeded far enough.

Let us mention that we computed between 20 and 50 eigenvalues in each case. Once $\lambda^+_{\max}$ was reached, all the remaining computed eigenvalues above it were about the same.

Similar results are obtained for our second experiment, on Panel B in Figure 6. Here, the entire plate is used, with the side AB perpendicular to the crack (which forms a 5 degree opening) now clamped, and the remaining boundary free. The loading is now

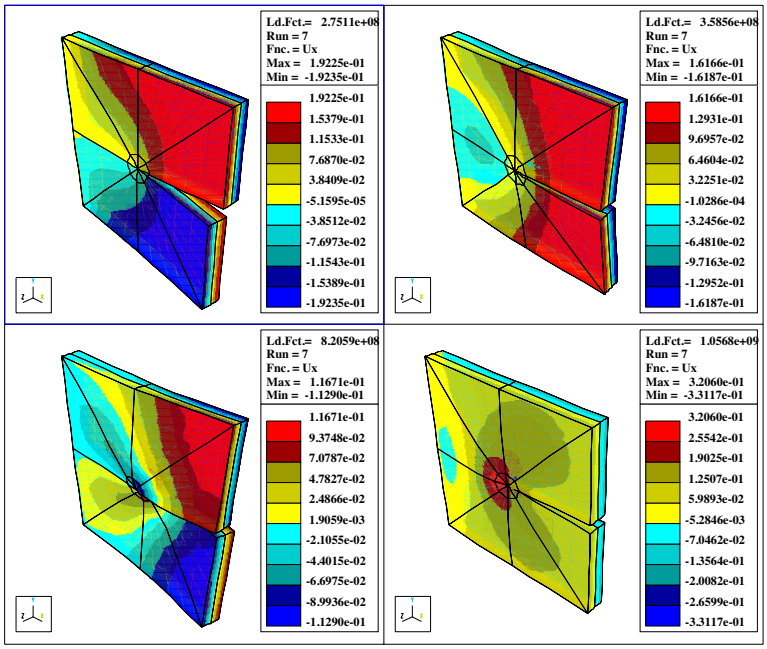

Figure 12: Computed eigenvectors for cracked panel with $d = 0.1$ and $n = 1$ layer

through a body force which is applied towards the midline throughout the body, i.e. -1 on the top half of the panel, and +1 on the lower half of the panel. Once again the strongest singularity occurs at the vertex of the crack, and we perform hp refinement with n layers. Spurious eigenvalues show up for much smaller d and n in this experiment, and in Figures 12-14 we show the first few eigenvectors this time for $d = 0.1$, with approximation degree $p = 8$ and $n = 1, 2$ and 3 layers, respectively (the eigenvalue convergence, not shown here, gives a graph similar to Figure 11).

For $n = 2$ the first two computed eigenvalues are physical, while the third one is spurious. Figure 13 shows the difference in the corresponding buckling modes. While the first two are global eigenvectors, the nature of the third is very local, with everything occurring close to the singularity. This is seen in the detail of an element from the innermost layer.

When we look at $n = 3$, all the eigenvectors are local, corresponding to spurious eigenvalues. Here, the refinement is already too strong, and $\lambda_{\max}(N)$ has been reached, with the result that none of the desired buckling eigenvalues can be computed. This shows that examination of the eigenvectors can be a useful *a posteriori* tool in determining whether a computed eigenvalue is physically relevant or not.

Remark 6.1 Let us point out that even in the example of the disc in Section 5, the stresses σ^*, if computed exactly, would be unbounded along edges. We do not see the influence of this unboundedness because the discretization is not strong enough near the

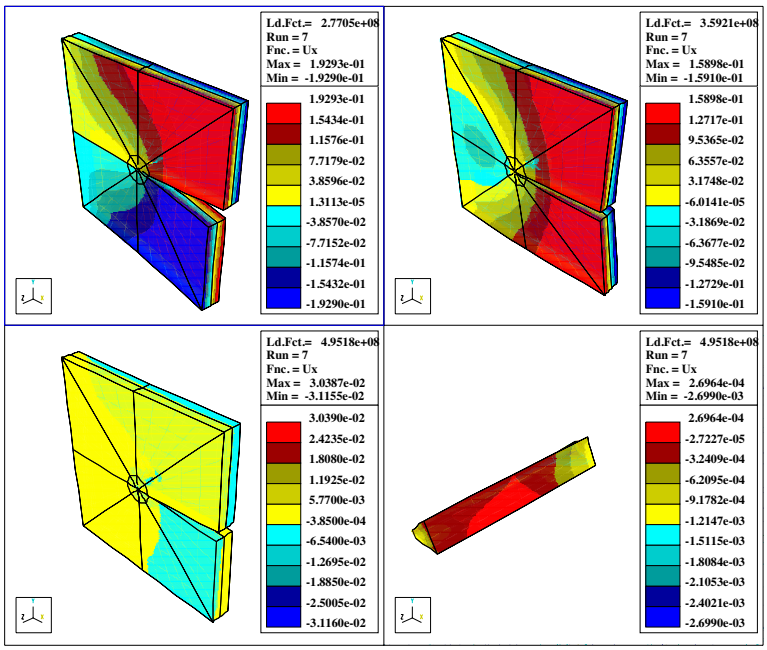

Figure 13: Computed eigenvectors for cracked panel with $d = 0.1$ and $n = 2$ layers

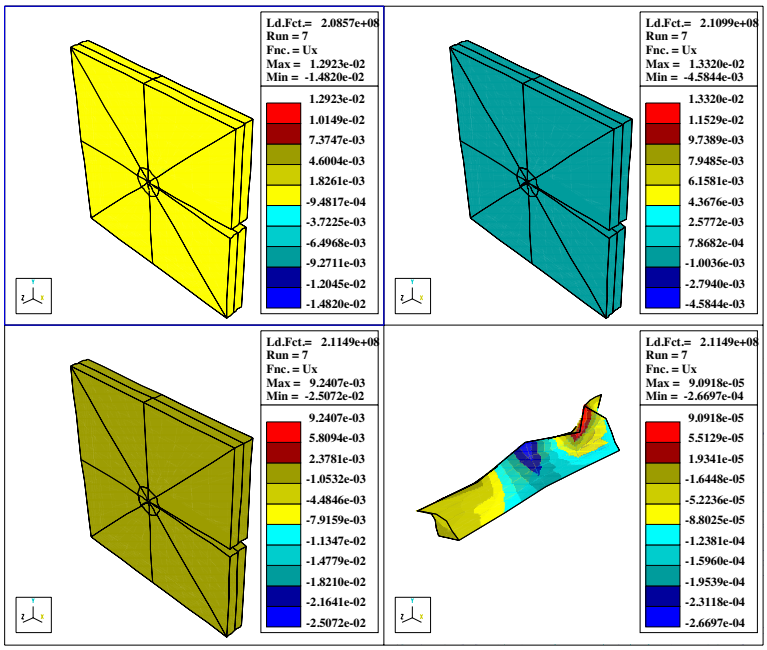

Figure 14: Computed eigenvectors for cracked panel with $d = 0.1$ and $n = 3$ layers

boundary to resolve the edge singularities. On the other hand, the pre-buckling stress σ^* is supposed to be representative of the typical physical stresses acting on the body, and it could be argued that the non-boundedness only arises because we use the linear theory (for convenience) to solve for it. However, similar problems occur even if the pre-buckling stress is bounded but with bound M_* (in equation (2.5)) large enough. In fact, in experiments where a reentrant corner (e.g. point O in Figure 10(b)) is replaced by a rounded fillet of very small radius, one observes that the phenomena discussed above remain essentially unchanged.

Acknowledgments

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