

Perturbation of the Cosserat spectrum

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Plan

- 1 **Cosserat spectrum**
- 2 4 avatars of the “first Cosserat eigenvalue”
 - avatar 1: Schur complement
 - avatar 2: LBB constant
 - avatar 3: Lifting of divergence
 - avatar 4: Friedrichs constant
- 3 Examples of perturbations
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Cosserat spectrum

Ω bounded domain in $\mathbb{R}^d$. Let $\mathcal{L}$ be the family of operators $\mathbb{R} \ni \sigma \mapsto L_\sigma$

$$\begin{aligned} L_\sigma : H_0^1(\Omega)^d &\longrightarrow H^{-1}(\Omega)^d \\ \mathbf{v} &\longmapsto \sigma \Delta \mathbf{v} - \nabla \operatorname{div} \mathbf{v} \end{aligned}$$

introduced by [Cosserat & Cosserat 1898] to solve Lamé system —

$$\mathbf{v} \mapsto \mu \Delta + (\lambda + \mu) \nabla \operatorname{div} \mathbf{v}$$

Spectrum of family $\mathcal{L}$

- Cosserat spectrum, $\mathfrak{S}(\mathcal{L})$: set of σ such that L_σ is not invertible,
- Essential spectrum, $\mathfrak{S}_{\text{ess}}(\mathcal{L})$: set of σ such that L_σ is not Fredholm.

Theorem [Michlin 1973] $d \geq 2$

- $0, 1 \in \mathfrak{S}_{\text{ess}}(\mathcal{L})$ because L_σ is not elliptic inside the domain
 - Any $\mathbf{v} = \operatorname{curl} \varphi$ (with $\varphi \in H_0^2(\Omega)$) belongs to $\ker L_0$
 - Any $\mathbf{v} = \nabla \varphi$ (with $\varphi \in H_0^2(\Omega)$) belongs to $\ker L_1$
- $\frac{1}{2} \in \mathfrak{S}_{\text{ess}}(\mathcal{L})$ because Dirichlet conditions do not cover L_σ
- If Ω is smooth $\mathfrak{S}_{\text{ess}}(\mathcal{L}) = \{0, \frac{1}{2}, 1\}$

Essential spectrum, polygonal domains

Ω (curvilinear) polygonal domain with one corner of opening ω .

- ① The theory of Kondrat'ev (revisited in [KozMazRos 1997]) $\implies$
 $\exists$ a **discrete set** Λ_σ of **singularity exponents** λ determining whether
 L_σ is Fredholm $H_0^1(\Omega)^2 \rightarrow H^{-1}(\Omega)^2$
- ② The explicit form of L_σ (generalized Lamé system) provides formulas
 for root functions of **singularity exponents** λ

Theorem [CoCrDaLa 2015] Let $\sigma \notin \{0, \frac{1}{2}, 1\}$

- The singularity exponents of L_σ are the roots $\lambda \in \mathbb{C}$ of equation

$$(\mathcal{D}) \quad (1 - 2\sigma) \frac{\sin \lambda \omega}{\lambda \omega} = \pm \frac{\sin \omega}{\omega}$$

- For real σ , roots of $(\mathcal{D})$ are real or purely imaginary.
- $L_\sigma : H_0^1(\Omega)^2 \rightarrow H^{-1}(\Omega)^2$ is Fredholm iff $(\mathcal{D})$ has no root s.t. **Re** $\lambda = 0$
- $\sigma \in \mathfrak{S}_{\text{ess}}(\mathcal{L}) \iff |1 - 2\sigma|\omega \leq |\sin \omega|$, i.e

$$\sigma \in \left[\frac{1}{2} - \frac{|\sin \omega|}{2\omega}, \frac{1}{2} + \frac{|\sin \omega|}{2\omega} \right]$$

Purely imaginary singularity exponents

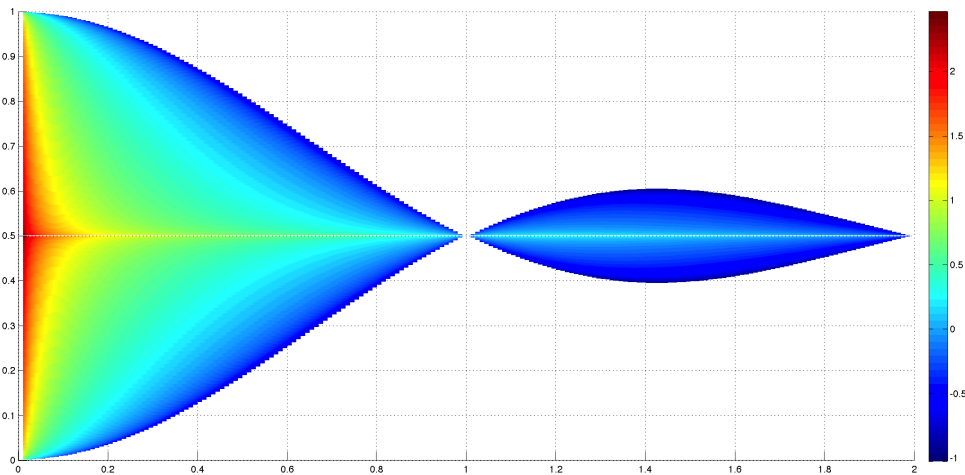


Figure: Color plot of the decimal logarithm of $|\text{Im } \lambda|$ for purely imaginary roots λ of the characteristic equation ($\mathcal{D}$) as a function of the opening $\omega/\pi \in (0, 2)$ (abscissa) and the Cosserat spectral parameter $\sigma \in [0, 1]$ (ordinate).

Questions of interest

The Cosserat spectrum $\mathfrak{S}(\mathcal{L})$ depends only on the **shape of the domain**:
 $\mathfrak{S}(\mathcal{L})$ is invariant by dilations, translations, symmetries and rotations.

We are interested in the bottom of the positive spectrum

$$\sigma_1(\mathcal{L}) = \inf (\mathfrak{S}(\mathcal{L}) \setminus \{0\})$$

Questions:

- Is it an eigenvalue?
- Bounds by above and below
- Continuity with respect to the domain, in which sense.
- Asymptotics

Elementary, but useful remark

$$\sigma_1(\mathcal{L}) \leq \inf (\mathfrak{S}_{\text{ess}}(\mathcal{L}) \setminus \{0\})$$

Other facts will be proved on the avatars of $\sigma_1(\mathcal{L})$...

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The Schur complement of the Stokes problem

$L^2_\circ(\Omega) = \{q \in L^2(\Omega) \text{ with } \int_\Omega q = 0\}$, and $\Delta^{-1} : H^{-1}(\Omega)^d \rightarrow H^1_0(\Omega)^d$

$$\begin{aligned} \mathcal{S} : L^2_\circ(\Omega) &\longrightarrow L^2_\circ(\Omega) \\ q &\longmapsto \operatorname{div} \Delta^{-1} \nabla q \end{aligned}$$

- $\mathcal{S} = \nabla' (-\Delta)^{-1} \nabla$, bounded, self-adjoint, non-negative.
- $\mathcal{S}$ operator of order 0. **Not compact, no compact resolvent.**
- Spectrum $\mathfrak{S}(\mathcal{S}) \subset [0, 1]$. **Essential spectrum $\mathfrak{S}_{\text{ess}}(\mathcal{S}) \neq \emptyset$**

$\sigma(\Omega)$ the 1st Rayleigh quotient of $\mathcal{S}$

$$\sigma(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \frac{\langle \mathcal{S}q, q \rangle_\Omega}{\langle q, q \rangle_\Omega}$$

Lemma

Relation of Cosserat spectrum with spectrum of Schur complement

$$\mathfrak{S}(\mathcal{L}) \setminus \{0\} = \mathfrak{S}(\mathcal{S}) \setminus \{0\} \quad \text{and} \quad \mathfrak{S}_{\text{ess}}(\mathcal{L}) \setminus \{0\} = \mathfrak{S}_{\text{ess}}(\mathcal{S}) \setminus \{0\}$$

$$\text{if } \sigma(\Omega) > 0$$

$$\sigma_1(\mathcal{L}) = \sigma(\Omega)$$

Explanation: Relation Cosserrat – Schur complement

Based on “classical” lemma

Lemma

Let two Banach spaces X and Y , and two bounded operators

$$A : Y \longrightarrow X \quad \text{and} \quad B : X \longrightarrow Y,$$

Let $\lambda \neq 0$. Then

- B bijective : $\ker(AB - \lambda\mathbb{I})$ onto $\ker(BA - \lambda\mathbb{I})$ with inverse A .
- B bijective : $X/(AB - \lambda\mathbb{I})(X)$ onto $Y/(BA - \lambda\mathbb{I})(Y)$ with inverse A .

$$\mathfrak{G}(AB) \setminus \{0\} = \mathfrak{G}(BA) \setminus \{0\} \quad \text{and} \quad \mathfrak{G}_{\text{ess}}(AB) \setminus \{0\} = \mathfrak{G}_{\text{ess}}(BA) \setminus \{0\}$$

Application: Recall $L_\sigma = \sigma\Delta - \nabla \operatorname{div}$. Introduce $\mathcal{T} = \Delta^{-1} \nabla \operatorname{div}$. Obvious

$$\mathfrak{G}(\mathcal{L}) = \mathfrak{G}(\mathcal{T}) \quad \text{and} \quad \mathfrak{G}_{\text{ess}}(\mathcal{L}) = \mathfrak{G}_{\text{ess}}(\mathcal{T})$$

Setting $X = L_0^2(\Omega)$, $Y = H_0^1(\Omega)$, $A = \operatorname{div}$ and $B = \Delta^{-1} \nabla$, we have

$$A : Y \longrightarrow X \quad \text{and} \quad B : X \longrightarrow Y, \quad \mathcal{S} = AB \quad \text{and} \quad \mathcal{T} = BA$$

Hence the wanted relations between $\mathcal{S}$ and $\mathcal{L}$



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The inf-sup constant, alias LBB constant

- $H_0^1(\Omega)$ sub-space of $u \in H^1(\Omega)$ with null trace on $\partial\Omega$. Poincaré inequ.
 $\rightarrow$ Semi-norm $|u|_{1,\Omega} := \|\nabla u\|_{0,\Omega}$ equivalent to norm $\|u\|_{1,\Omega}$

LBB constant, after Ladyzhenskaya, Babuška and Brezzi

$$\beta(\Omega) = \inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{\int_{\Omega} \operatorname{div} \mathbf{v} \, q}{|\mathbf{v}|_{1,\Omega} \|q\|_{0,\Omega}}$$

- If $d = 1$, $\beta(\Omega) = 1$:
 Ω is an interval (a, b) . For $q \in L_0^2(\Omega)$, take $v(x) = \int_a^x q(t) \, dt$.
- In any dimension $\beta(\Omega) \geq 0$
- In any dimension $\beta(\Omega) \leq 1$, because of the identity

$$\forall \mathbf{v} \in H_0^1(\Omega)^d, \quad |\mathbf{v}|_{1,\Omega}^2 = \|\nabla \mathbf{v}\|_{0,\Omega}^2 = \|\overrightarrow{\operatorname{curl}} \mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div} \mathbf{v}\|_{0,\Omega}^2$$

- $\beta(\Omega)$ is invariant by dilations, translations, symmetries and rotations.
 Depends only on the *shape* of Ω

Inf-sup constant and duality

$H^{-1}(\Omega)$ dual space of $H_0^1(\Omega)$ and $H^{-1}(\Omega)^d$ dual space of $H_0^1(\Omega)^d$.

Dual norm associated with the norm $|\cdot|_{1,\Omega}$ in $H_0^1(\Omega)^d$:

$$\text{For any } \varphi \in H^{-1}(\Omega)^d, \quad |\varphi|_{-1,\Omega} = \sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{\langle \varphi, \mathbf{v} \rangle_\Omega}{|\mathbf{v}|_{1,\Omega}}$$

And for $\varphi = \nabla q$:

$$|\nabla q|_{-1,\Omega} = \sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{\langle \nabla q, \mathbf{v} \rangle_\Omega}{|\mathbf{v}|_{1,\Omega}} = \sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{\int_\Omega \operatorname{div} \mathbf{v} q}{|\mathbf{v}|_{1,\Omega}}$$

The inf-sup constant and the dual norm

$$[\mathbf{B}] \quad \beta(\Omega) = \inf_{q \in L_0^2(\Omega)} \frac{|\nabla q|_{-1,\Omega}}{\|q\|_{0,\Omega}}$$

Relation of LBB with Schur complement $\mathcal{S}$

Lemma

$$\sigma(\Omega) = \beta(\Omega)^2$$

Proof: Δ realizes a Riesz isometry between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, thus between $H_0^1(\Omega)^d$ and $H^{-1}(\Omega)^d$:

$$\forall \mathbf{v} \in H_0^1(\Omega)^d, \quad \langle -\Delta \mathbf{v}, \mathbf{v} \rangle_{H^{-1}(\Omega)^d, H_0^1(\Omega)^d} = |\mathbf{v}|_{1,\Omega}^2 = |\Delta \mathbf{v}|_{-1,\Omega}^2$$

Let $q \in L_0^2(\Omega)$ (and set $\mathbf{v} = \Delta^{-1} \nabla q$ to help)

$$\langle \mathcal{S} q, q \rangle_\Omega = \langle \operatorname{div} \Delta^{-1} \nabla q, q \rangle_\Omega = -\langle \nabla q, \Delta^{-1} \nabla q \rangle_{H^{-1}(\Omega)^d, H_0^1(\Omega)^d} = |\nabla q|_{-1,\Omega}^2$$

Hence

$$\sigma(\Omega) = \inf_{q \in L_0^2(\Omega)} \frac{\langle \mathcal{S} q, q \rangle_\Omega}{\langle q, q \rangle_\Omega} = \inf_{q \in L_0^2(\Omega)} \frac{|\nabla q|_{-1,\Omega}^2}{\|q\|_\Omega^2} \stackrel{[B]}{=} \beta(\Omega)^2$$

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Lifting of divergence

The Babuška–Aziz constant $C(\Omega)$

$C(\Omega) \in \mathbb{R} \cup \{\infty\}$ is the smallest number C such that $\forall q \in L^2_0(\Omega)$

$$\inf_{\mathbf{v} \in H_0^1(\Omega)^d \text{ with } \operatorname{div} \mathbf{v} = q} |\mathbf{v}|_{1,\Omega}^2 \leq C \|q\|_{0,\Omega}^2$$

with convention $C(\Omega) = \infty$ if $\operatorname{div} \mathbf{v} = q$ has no solution in $H_0^1(\Omega)^d$.

Lemma

The constant $C(\Omega)$ is finite if and only if $\beta(\Omega)$ is positive and there holds,

$$C(\Omega) = \frac{1}{\beta(\Omega)^2} = \frac{1}{\sigma(\Omega)} = \frac{1}{\sigma_1(\mathcal{L})}$$

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Friedrichs constant

- $\Omega \subset \mathbb{R}^2 \sim \mathbb{C}$ and $\mathfrak{F}(\Omega)$ space of complex valued $L^2(\Omega)$ holomorphic functions. $\mathfrak{F}_0(\Omega)$ the subspace of $\mathfrak{F}(\Omega)$ with mean value 0.
- Let $h, g \in L^2(\Omega)$ with real values. h is *harmonic conjugate* to g if the function $(x_1 + i x_2) \mapsto h(x_1, x_2) + i g(x_1, x_2)$ is *holomorphic*.

The Friedrichs constant $\Gamma(\Omega)$

$\Gamma(\Omega) \in \mathbb{R} \cup \{\infty\}$ is the smallest constant Γ such that $\forall h + ig \in \mathfrak{F}_0(\Omega)$

$$\|h\|_{L^2(\Omega)}^2 \leq \Gamma \|g\|_{L^2(\Omega)}^2$$

Theorem [Horgan-Payne 1983] [Costabel-Dauge 2015]

There hold the equivalence

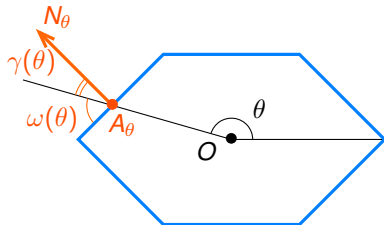
$$\beta(\Omega) > 0 \iff \Gamma(\Omega) < \infty$$

and the identity

$$\Gamma(\Omega) + 1 = \frac{1}{\beta(\Omega)^2} = C(\Omega) = \frac{1}{\sigma(\Omega)} = \frac{1}{\sigma_1(\mathcal{L})}$$

Horgan-Payne lower bound

- **Domain** $\Omega \subset \mathbb{R}^2$ star-shaped with respect to point O
- **Angle** $\gamma(\theta)$ between line $[O, A_\theta]$ and N_θ , and $\omega(\theta) = \frac{\pi}{2} - \gamma(\theta)$.



HP-angle (introduced by Stoyan)

$$\omega_{HP}(\Omega) := \frac{\pi}{2} - \sup_{\theta \in \mathbb{T}} \gamma(\theta) = \inf_{\theta \in \mathbb{T}} \omega(\theta)$$

Theorem [Horgan-Payne 1983] & [Costabel-Dauge 2015]

If Ω is a convex polygon with concentric corners

$$\beta(\Omega) \geq \sin \frac{\omega_{HP}(\Omega)}{2}$$

Examples

- Triangles
- Rectangles
- Regular convex polygons
- Disc (!) : $\beta(\Omega) = \frac{1}{\sqrt{2}}$

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Ex1: Regular convex polygons tending to a disc

Let, in dimension $d = 2$

- Ω_N “the” regular convex polygon with N sides, $N \geq 3$
- Ω “the” disc

By result on essential spectrum

$$\beta(\Omega_N)^2 \leq \frac{1}{2}.$$

The HP angle of Ω_N

$$\omega_{HP}(\Omega_N) = \frac{\pi}{2} - \frac{\pi}{N}$$

Hence lower bound

$$\sin\left(\frac{\pi}{4} - \frac{\pi}{2N}\right) \leq \beta(\Omega_N)$$

Using $\beta(\Omega)^2 = \frac{1}{2}$,

$$0 < \beta(\Omega) - \beta(\Omega_N) \leq \frac{\pi}{2N}$$

Convergence !

Ex2: Rings tending to a disc

Let, in dimension $d = 2$

- Ω_N the ring with inner radius $\frac{1}{N}$ and outer radius 1, $N \geq 2$
- Ω the unit disc

Explicit formula [Chizhonkov Olshanskii 2000] + [BeCoDaGi 20??]

$$\beta(\Omega_N)^2 = \frac{1}{2} \left(1 - \sqrt{\frac{N^2 - 1}{N^2 + 1} \frac{1}{\log N}} \right)$$

Hence

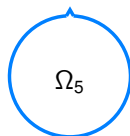
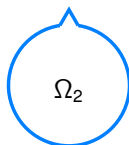
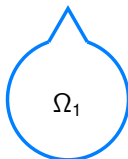
$$0 < \beta(\Omega)^2 - \beta(\Omega_N)^2 \leq \frac{1}{\sqrt{\log N}}$$

Convergence !

Ex3: Disc with small hats tending to a disc

Let, in dimension $d = 2$

- Ω_N the union of the unit disc and an equilateral triangle with side length $\frac{1}{N}$ and top vertex at $(0, 1 + \frac{1}{2N})$, for $N \geq 1$
- Ω the unit disc



Upper bound (essential spectrum due to angle $\omega = \frac{\pi}{3}$):

$$\beta(\Omega_N)^2 \leq \frac{1}{2} - \frac{\sin \omega}{2\omega} \quad \text{with} \quad \omega = \frac{\pi}{3}, \quad \text{for all } N \geq 1,$$

Hence

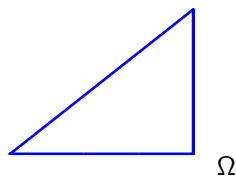
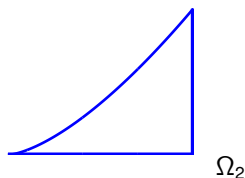
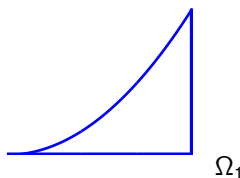
$$\beta(\Omega)^2 - \beta(\Omega_N)^2 \geq \frac{3\sqrt{3}}{4\pi}$$

Non-convergence !

Ex4: Cusp domains tending to a triangle

Let, in dimension $d = 2$

- Ω_N the curvilinear triangular domain with sides $y = 0$, $x = 1$, and $y = x^{1+\frac{1}{N}}$, for $N \geq 1$
- Ω the triangle with sides $y = 0$, $x = 1$, and $y = x$.



For all $N \geq 1$, Ω_N is a cusp domain, hence $\beta(\Omega_N) = 0$ [Tartar 2006].

Ω is a triangle and HP Theorem provides a positive lower bound for $\beta(\Omega)$

Non-convergence !

Upper semi-continuity

Theorem [BeCoDaGi 2015]

Let us assume that the sequence of domains $(\Omega_N)_{N \geq 1}$ converges to the limiting domain Ω in the following sense

$$\Omega_N \subset \Omega, \forall N \geq 1 \quad \text{and} \quad \text{meas}(\Omega \setminus \Omega_N) \xrightarrow[N \rightarrow \infty]{} 0$$

Then there holds

$$\limsup_{N \rightarrow \infty} \beta(\Omega_N) \leq \beta(\Omega)$$

Examples

ex3 $\beta(\Omega_N)^2 \leq \frac{1}{2} - \frac{3\sqrt{3}}{4\pi}$ and $\beta(\Omega)^2 = \frac{1}{2}$

ex4 $\beta(\Omega_N) = 0$ and $\beta(\Omega) > 0$.

Continuity

Two domains Ω and $\tilde{\Omega}$ linked by a bijective mapping $\mathcal{F}$

$$\mathcal{F}(\Omega) = \tilde{\Omega}.$$

Theorem [Zsuppán 2011]

$\Omega, \tilde{\Omega}$ simply connected domains in $\mathbb{R}^2$ with piecewise smooth boundary.
Let g be the bijective conformal mapping of Ω onto $\tilde{\Omega}$.

$$\max_{x \in \tilde{\Omega}} |g' - 1| := \varepsilon < 1 \quad \implies \quad |\beta(\Omega) - \beta(\tilde{\Omega})| \leq \frac{\sqrt{2} \varepsilon}{1 - \varepsilon}$$

Theorem [BeCoDaGi 2015]

$\Omega, \tilde{\Omega}$ Lipschitz bounded domains in $\mathbb{R}^d$.

Assume that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Lipschitz. Identity denoted $\mathcal{I}_d$.

There exists a constant $c(d)$ depending only on d s.t.

$$\begin{cases} \|D\mathcal{F} - \mathcal{I}_d\|_{L^\infty(\Omega)} \leq \varepsilon \\ \|D\mathcal{F}^{-1} - \mathcal{I}_d\|_{L^\infty(\tilde{\Omega})} \leq \varepsilon \end{cases} \implies |\beta(\Omega) - \beta(\tilde{\Omega})| \leq c(d) \varepsilon$$

Piecewise linear approximation of boundaries

Let $\mathfrak{C}(\Omega)$ the set of corners of a polygonal domain. If Ω smooth, $\mathfrak{C}(\Omega) = \emptyset$.

$\Omega \subset \mathbb{R}^2$ polygonal domain and $h > 0$.

A *polygonal h -approximation* of Ω is a straight polygon Ω_h such that

- (i) $\mathfrak{C}(\Omega) \subset \mathfrak{C}(\Omega_h)$,
- (ii) $\mathfrak{C}(\Omega_h) \subset \partial\Omega$,
- (iii) The length of each side of Ω_h is less than h .

Corollary of continuity theorem:

Theorem [BeCoDaGi 2015]

$\Omega \subset \mathbb{R}^2$ Lipschitz polygonal domain with piecewise $\mathcal{C}^2$ -boundary.

$\exists$ constant $c(\Omega)$ such that for all h -approximation Ω_h of Ω

$$|\beta(\Omega) - \beta(\Omega_h)| \leq c(\Omega) h$$

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Product domains (upper bounds)

Here $\Omega = A \times B$. Coordinates $x \in A, y \in B$.

Theorem

$$\sigma(\Omega) \leq \min\{\sigma(A), \sigma(B)\}$$

Not informative if A and B are intervals (then $\sigma(A) = \sigma(B) = 1$).

Theorem

- λ_A^{Neu} the 2d Laplace-Neumann eigenvalue of A (1st is 0)
- 1_B the constant function 1 on B

$$\sigma(\Omega) \leq \lambda_A^{\text{Neu}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)}$$

Remark Swap the roles of A and B

$$\sigma(\Omega) \leq \frac{\langle (-\Delta_x)^{-1} 1_A, 1_A \rangle_A}{\text{meas}(A)} \lambda_B^{\text{Neu}}$$

Application to rectangles (upper bounds)

$A = (-a, a)$ and $B = (-b, b)$. Aspect ratio $\rho = \min\{\frac{a}{b}, \frac{b}{a}\}$

$$\lambda_A^{\text{Neu}} = \frac{\pi^2}{4a^2} \quad \text{and} \quad \lambda_B^{\text{Neu}} = \frac{\pi^2}{4b^2}$$

Moreover

$$(-\Delta_x)^{-1}1_A = \frac{a^2}{2} - \frac{x^2}{2} \quad \text{and} \quad (-\Delta_y)^{-1}1_B = \frac{b^2}{2} - \frac{y^2}{2}$$

Thus

$$\frac{\langle (-\Delta_x)^{-1}1_A, 1_A \rangle_A}{\text{meas}(A)} = \frac{a^2}{3} \quad \text{and} \quad \frac{\langle (-\Delta_y)^{-1}1_B, 1_B \rangle_B}{\text{meas}(B)} = \frac{b^2}{3}$$

Theorem

$$\sigma((-a, a) \times (-b, b)) \leq \min \left\{ \frac{\pi^2}{12} \frac{a^2}{b^2}, \frac{\pi^2}{12} \frac{b^2}{a^2} \right\} = \frac{\pi^2}{12} \rho^2$$

Application to thin domains (upper bounds)

Choose A_1 and B_1 . Consider an associated family Ω_ε of *thin domains*

$$\varepsilon \in (0, 1], \quad \Omega_\varepsilon = A \times B = A_1 \times (\varepsilon B_1)$$

Since β and σ are dilatation invariant, it is equivalent to consider

$$\varepsilon \in (0, 1], \quad \Omega_\varepsilon = A \times B = (\varepsilon^{-1} A_1) \times B_1$$

Since

$$\lambda_A^{\text{Neu}} = \varepsilon^2 \lambda_{A_1}^{\text{Neu}}$$

we find

$$\sigma(\Omega_\varepsilon) \leq \varepsilon^2 \lambda_{A_1}^{\text{Neu}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)}$$

Asymptotic lower bound for thin domains

Family of thin domains

$$\varepsilon \in (0, 1], \quad \Omega_\varepsilon = A \times B = (\varepsilon^{-1} A_1) \times B_1$$

New theorem

$\exists$ constant $K = K(A_1, B_1)$ depending only on A_1 and B_1 so that $\forall \varepsilon \in (0, 1]$

$$\sigma(\Omega_\varepsilon) \geq \varepsilon^2 \sigma(A_1) \lambda_{A_1}^{\text{Dir}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)} - K(A_1, B_1) \varepsilon^3$$

Compare with upper bound

$$\sigma(\Omega_\varepsilon) \leq \varepsilon^2 \lambda_{A_1}^{\text{Neu}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)}$$

Asymptotic lower bound for thin domains

Family of thin domains

$$\varepsilon \in (0, 1], \quad \Omega_\varepsilon = A \times B = (\varepsilon^{-1} A_1) \times B_1$$

New theorem

$\exists$ constant $K = K(A_1, B_1)$ depending only on A_1 and B_1 so that $\forall \varepsilon \in (0, 1]$

$$\sigma(\Omega_\varepsilon) \geq \varepsilon^2 \left[\sigma(A_1) \lambda_{A_1}^{\text{Dir}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)} - K(A_1, B_1) \varepsilon^3 \right]$$

Compare with upper bound

$$\sigma(\Omega_\varepsilon) \leq \varepsilon^2 \left[\lambda_{A_1}^{\text{Neu}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)} \right]$$

Remark: $\lambda_{A_1}^{\text{Dir}} \geq \lambda_{A_1}^{\text{Neu}}$ [Filonov 2005]

Asymptotics for channel domains

Channel domain = Thin domain with A_1 an interval, say $(0, 1)$.

Consequences

- $\sigma(A_1) = 1$ and $\lambda_{A_1}^{\text{Dir}} = \lambda_{A_1}^{\text{Neu}}$.

- $\sigma(\Omega_\varepsilon) = \varepsilon^2 \lambda_{A_1}^{\text{Neu}} \frac{\langle (-\Delta_y)^{-1} 1_B, 1_B \rangle_B}{\text{meas}(B)} + \mathcal{O}(\varepsilon^3)$

- Application to rectangles: $A = (0, \varepsilon^{-1})$, $B = (0, 1)$

$$\sigma(\Omega_\varepsilon) = \varepsilon^2 \frac{\pi^2}{12} + \mathcal{O}(\varepsilon^3)$$

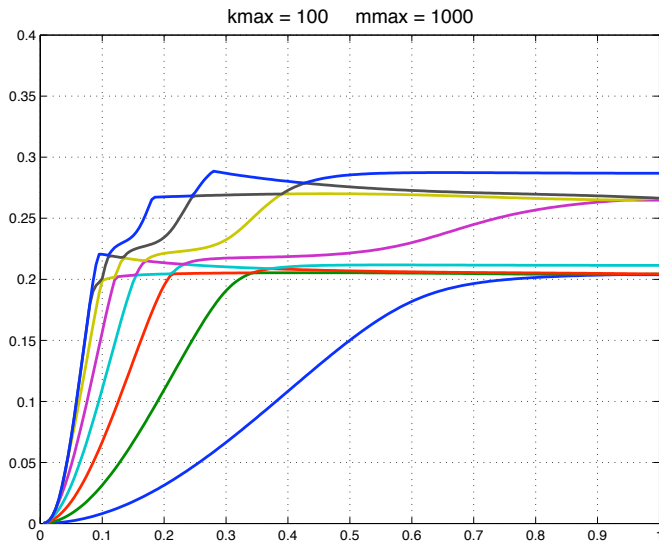
- Infimum of essential spectrum

$$\inf \mathfrak{S}_{\text{ess}}(\mathcal{S}) = \frac{1}{2} - \frac{|\sin \omega|}{2\omega} = \frac{1}{2} - \frac{1}{\pi} \simeq 0.1817$$

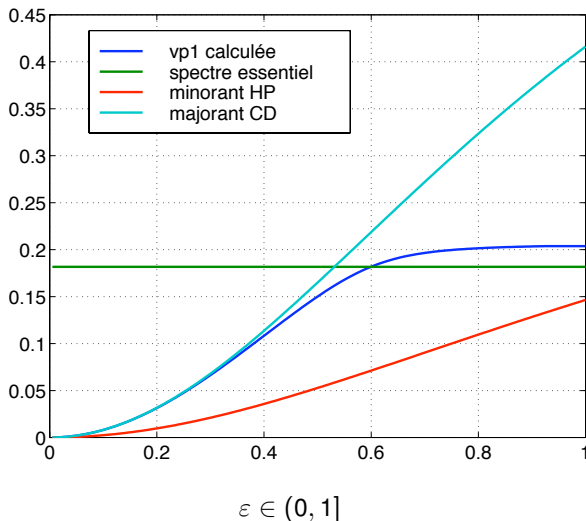
- HP angle $\omega_{HP}(\Omega_\varepsilon) = \arctan \varepsilon$. Hence the rough lower bound

$$\sigma(\Omega) \geq \sin \left(\frac{\arctan \varepsilon}{2} \right)^2 \underset{\rho \rightarrow 0}{\simeq} \frac{\varepsilon^2}{4}$$

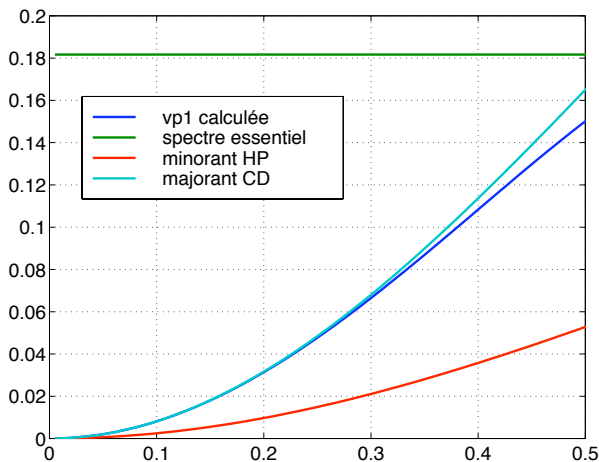
First 8 eigenval. of $\mathcal{S}$ in $(0, \varepsilon^{-1}) \times (0, 1)$, function of ε



First eigenval. vs lower (HP) & upper bounds

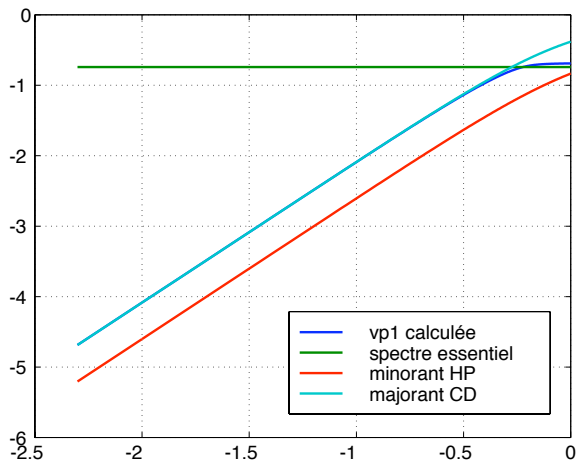


First eigenval. vs lower (HP) & upper bounds



Zoom: $\varepsilon \in (0, 0.5]$

First eigenval. vs lower (HP) & upper bounds



Log10-Log10 scale for $\varepsilon \in [0.005, 1]$

Plan

- 1 Cosserat spectrum
- 2 4 avatars of the “first Cosserat eigenvalue”
 - avatar 1: Schur complement
 - avatar 2: LBB constant
 - avatar 3: Lifting of divergence
 - avatar 4: Friedrichs constant
- 3 Examples of perturbations
- 4 Products of domains
- 5 References

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