

Edge stress intensity functions in 3-D anisotropic composites

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Abstract

The elastic solution in the vicinity of a traction free edge and an interface crack in cross-ply anisotropic laminates is provided in details for three problems. Complex eigen-pairs and edge stress intensity functions are computed with high accuracy, aimed to serve as benchmark problems against which other numerical methods can be checked.

Keywords: Edge stress intensity functions, p-FEM, composite materials, fracture mechanics, complex stress intensity factors.

1 Introduction

Delamination starting at a traction free edge or interface cracks between plies lead most frequently to catastrophic failures in cross-ply laminates. Many of these laminates are made of brittle anisotropic materials and occupy three-dimensional (3-D) domains, therefore the 3-D elastic solution in the vicinity of these edges (a crack front is particular case of a general edge) is of major engineering interest. The mathematical complexity of the 3-D anisotropic elasticity system in the vicinity of an edge makes it intractable by analytical methods and very complex even for numerical treatment. Therefore most of past studies reduced the problem to two-dimensions addressing the solution in the vicinity of singular points, see e.g. [4,6]. Only very limited studies considered restricted classes of 3-D edge singularity

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problems, as [2] for instance which investigates the specific through interface crack between a $\pm 45^\circ$ transversely isotropic pair of materials.

Recently, the explicit mathematical representation of the 3-D elastic solution in the vicinity of a straight edge has been provided and the quasi-dual function method was proposed for the extraction of edge stress intensity function (ESIF) [1]. This enabled the development of highly accurate numerical methods [8,7,3] used herein for the computation of elastic solutions for three benchmark problems including the full expansion in the vicinity of a traction free edge and interface crack. The problems are suggested to be used as benchmarks against which methods for the computation of edge stress intensity functions can be compared.

2 The elastic solution in the vicinity of an edge

Consider a 3-D domain in which a straight edge is of interest (see Figure 1 for example). A cylindrical coordinate system (r, θ, x_3) is located at a point on the edge so that x_3 is along the edge. The two flat planes that intersect at the edge are denoted by Γ_1 and Γ_2 . The displacements vector and strain and stress tensors are represented in cylindrical coordinates: $\mathbf{u} = \{u_r, u_\theta, u_{x_3}\}^T$, $\boldsymbol{\varepsilon} = \{\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{33}, \gamma_{\theta 3}, \gamma_{r 3}, \gamma_{r\theta}\}^T$ and $\boldsymbol{\sigma} = \{\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{33}, \sigma_{\theta 3}, \sigma_{r 3}, \sigma_{r\theta}\}^T$. The linear elasticity problem is being solved by means of separation of variables, yielding in an elastic solution that can be presented in the vicinity of an edge by the series [1,8]:

$$\mathbf{u} = \sum_{i \geq 1} \sum_{j \geq 0} \partial_3^j A_i(x_3) \Phi_j^{(\alpha_i)}(r, \theta), \quad \Phi_j^{(\alpha_i)}(r, \theta) = r^{\alpha_i+j} \varphi_j^{(\alpha_i)}(\theta) \quad (1)$$

or in a more explicit form:

$$\begin{aligned} \mathbf{u} = & A_1(x_3) r^{\alpha_1} \varphi_0^{(\alpha_1)}(\theta) + \partial_3 A_1(x_3) r^{\alpha_1+1} \varphi_1^{(\alpha_1)}(\theta) + \partial_3^2 A_1(x_3) r^{\alpha_1+2} \varphi_2^{(\alpha_1)}(\theta) + \dots \\ & + A_2(x_3) r^{\alpha_2} \varphi_0^{(\alpha_2)}(\theta) + \partial_3 A_2(x_3) r^{\alpha_2+1} \varphi_1^{(\alpha_2)}(\theta) + \partial_3^2 A_2(x_3) r^{\alpha_2+2} \varphi_2^{(\alpha_2)}(\theta) + \dots \\ & + A_3(x_3) r^{\alpha_3} \varphi_0^{(\alpha_3)}(\theta) + \partial_3 A_3(x_3) r^{\alpha_3+1} \varphi_1^{(\alpha_3)}(\theta) + \partial_3^2 A_3(x_3) r^{\alpha_3+2} \varphi_2^{(\alpha_3)}(\theta) + \dots \\ & + \dots \end{aligned} \quad (2)$$

where $A_i(x_3)$ is the Edge Stress Intensity Function (ESIF) associated with the i^{th} eigen-pair, the *singular function* $\Phi_0^{(\alpha_i)} = r^{\alpha_i} \varphi_0^{(\alpha_i)}(\theta)$ being the well known two-dimensional eigen-function, whereas $\Phi_j^{(\alpha_i)} = r^{\alpha_i+j} \varphi_j^{(\alpha_i)}(\theta)$, $j \geq 1$ are the *shadow functions* of the primal singular function. The angular part of eigen-functions and their shadows have three components denoted by:

$$\varphi_j^{(\alpha_i)}(\theta) = \left(\varphi_{jr}^{(\alpha_i)}(\theta), \varphi_{j\theta}^{(\alpha_i)}(\theta), \varphi_{jx_3}^{(\alpha_i)}(\theta) \right)^T, \quad j = 0, 1, \dots$$

Herein any eigen-function $\left(\varphi_{jr}^{(\alpha_i)}(\theta), \varphi_{j\theta}^{(\alpha_i)}(\theta), \varphi_{jx_3}^{(\alpha_i)}(\theta) \right)^T$ or any shadow-function are expressed by sin, cos series. For example, if $\varphi_{jr}^{(\alpha_i)}(\theta)$ is to be evaluated then

$$\begin{aligned}
\varphi_{jr}^{(\alpha_i)} = & C_1 \sin(\alpha_i \theta) + C_2 \sin((\alpha_i + 1)\theta) + C_3 \sin((\alpha_i - 1)\theta) \\
& + C_4 \sin((\alpha_i + 2)\theta) + C_5 \sin((\alpha_i - 2)\theta) + C_6 \sin((\alpha_i + 3)\theta) + C_7 \sin((\alpha_i - 3)\theta) \\
& + C_8 \cos(\alpha_i \theta) + C_9 \cos((\alpha_i + 1)\theta) + C_{10} \cos((\alpha_i - 1)\theta) \\
& + C_{11} \cos((\alpha_i + 2)\theta) + C_{12} \cos((\alpha_i - 2)\theta) + C_{13} \cos((\alpha_i + 3)\theta) + C_{14} \cos((\alpha_i - 3)\theta)
\end{aligned} \tag{3}$$

There are cases for which an eigen-value is complex, i.e. $\alpha_i = \alpha_{i\Re} + i\alpha_{i\Im}$ and so is its corresponding eigenfunction and shadows $\varphi_j^{(\alpha_i)}(\theta) = \varphi_{j\Re}^{(\alpha_i)}(\theta) + i\varphi_{j\Im}^{(\alpha_i)}(\theta)$ are complex, therefore the corresponding ESIF is complex also $A_i(x_3) = A_{i\Re}(x_3) + iA_{i\Im}(x_3)$. In this case $\varphi_{j\Re}^{(\alpha_i)}(\theta), \varphi_{j\Im}^{(\alpha_i)}(\theta)$ are obtained by substituting $\alpha_{i\Re}$ and $\alpha_{i\Im}$ in (3) respectively.

For complex eigen-pairs, $\alpha_{i+1} = \alpha_{i\Re} - i\alpha_{i\Im}$, $\varphi_j^{(\alpha_{i+1})}(\theta) = \varphi_{j\Re}^{(\alpha_i)}(\theta) - i\varphi_{j\Im}^{(\alpha_i)}(\theta)$ and $A_{i+1}(x_3) = A_{i\Re}(x_3) - iA_{i\Im}(x_3)$. For a crack in a bi-material interface for example, the first eigen-value is complex $\alpha_{1,\Re} \pm i\alpha_{1,\Im}$, so that (2) is:

$$\begin{aligned}
\mathbf{u} = & (A_{1\Re}(x_3) + iA_{1\Im}(x_3)) r^{\alpha_{1\Re} + i\alpha_{1\Im}} \left(\varphi_{0\Re}^{(\alpha_1)}(\theta) + \varphi_{0\Im}^{(\alpha_1)}(\theta) \right) \\
& + (\partial_3 A_{1\Re}(x_3) + i\partial_3 A_{1\Im}(x_3)) r^{1 + \alpha_{1\Re} + i\alpha_{1\Im}} \left(\varphi_{1\Re}^{(\alpha_1)}(\theta) + \varphi_{1\Im}^{(\alpha_1)}(\theta) \right) \\
& + \left(\partial_3^2 A_{1\Re}(x_3) + i\partial_3^2 A_{1\Im}(x_3) \right) r^{2 + \alpha_{1\Re} + i\alpha_{1\Im}} \left(\varphi_{2\Re}^{(\alpha_1)}(\theta) + \varphi_{2\Im}^{(\alpha_1)}(\theta) \right) + \dots \\
& + (A_{1\Re}(x_3) - iA_{1\Im}(x_3)) r^{\alpha_{1\Re} - i\alpha_{1\Im}} \left(\varphi_{0\Re}^{(\alpha_1)}(\theta) - \varphi_{0\Im}^{(\alpha_1)}(\theta) \right) \\
& + (\partial_3 A_{1\Re}(x_3) - i\partial_3 A_{1\Im}(x_3)) r^{1 + \alpha_{1\Re} - i\alpha_{1\Im}} \left(\varphi_{1\Re}^{(\alpha_1)}(\theta) - \varphi_{1\Im}^{(\alpha_1)}(\theta) \right) \\
& + \left(\partial_3^2 A_{1\Re}(x_3) - i\partial_3^2 A_{1\Im}(x_3) \right) r^{2 + \alpha_{1\Re} - i\alpha_{1\Im}} \left(\varphi_{2\Re}^{(\alpha_1)}(\theta) - \varphi_{2\Im}^{(\alpha_1)}(\theta) \right) + \dots \\
& + A_3(x_3) r^{\alpha_3} \varphi_0^{(\alpha_3)}(\theta) + \partial_3 A_3(x_3) r^{\alpha_3 + 1} \varphi_1^{(\alpha_3)}(\theta) + \partial_3^2 A_3(x_3) r^{\alpha_3 + 2} \varphi_2^{(\alpha_3)}(\theta) + \dots
\end{aligned} \tag{4}$$

A straight forward algebraic manipulation allows to represent (4) without complex numbers:

$$\begin{aligned}
\mathbf{u} = & 2r^{\alpha_{1\Re}} \cos(\alpha_{1\Im} \ln r) \left(A_{1\Re}(x_3) \varphi_{0\Re}^{(\alpha_1)}(\theta) - A_{1\Im}(x_3) \varphi_{0\Im}^{(\alpha_1)}(\theta) \right) \\
& - 2r^{\alpha_{1\Re}} \sin(\alpha_{1\Im} \ln r) \left(A_{1\Im}(x_3) \varphi_{0\Re}^{(\alpha_1)}(\theta) + A_{1\Re}(x_3) \varphi_{0\Im}^{(\alpha_1)}(\theta) \right) \\
& + 2r^{1 + \alpha_{1\Re}} \cos(\alpha_{1\Im} \ln r) \left(\partial_3 A_{1\Re}(x_3) \varphi_{1\Re}^{(\alpha_1)}(\theta) - \partial_3 A_{1\Im}(x_3) \varphi_{1\Im}^{(\alpha_1)}(\theta) \right) \\
& - 2r^{1 + \alpha_{1\Re}} \sin(\alpha_{1\Im} \ln r) \left(\partial_3 A_{1\Im}(x_3) \varphi_{1\Re}^{(\alpha_1)}(\theta) + \partial_3 A_{1\Re}(x_3) \varphi_{1\Im}^{(\alpha_1)}(\theta) \right) \\
& + 2r^{2 + \alpha_{1\Re}} \cos(\alpha_{1\Im} \ln r) \left(\partial_3^2 A_{1\Re}(x_3) \varphi_{2\Re}^{(\alpha_1)}(\theta) - \partial_3^2 A_{1\Im}(x_3) \varphi_{2\Im}^{(\alpha_1)}(\theta) \right) \\
& - 2r^{2 + \alpha_{1\Re}} \sin(\alpha_{1\Im} \ln r) \left(\partial_3^2 A_{1\Im}(x_3) \varphi_{2\Re}^{(\alpha_1)}(\theta) + \partial_3^2 A_{1\Re}(x_3) \varphi_{2\Im}^{(\alpha_1)}(\theta) \right) + \dots \\
& + A_3(x_3) r^{\alpha_3} \varphi_0^{(\alpha_3)}(\theta) + \partial_3 A_3(x_3) r^{\alpha_3 + 1} \varphi_1^{(\alpha_3)}(\theta) + \partial_3^2 A_3(x_3) r^{\alpha_3 + 2} \varphi_2^{(\alpha_3)}(\theta) + \dots
\end{aligned} \tag{5}$$

3 Benchmark Problems

p -Finite element methods have been utilized for the solution of three benchmark problems: The first is a crack at a bi-material interface between two isotropic

materials, the second is a crack in a compact test specimen at an interface of two anisotropic materials, and the third is an edge at the intersection of two anisotropic materials and flux free face. These solutions, including the eigen-pairs, shadow functions and ESIFs are provided in this section.

3.1 Benchmark problem A - Crack at the interface of two isotropic materials

A bi-material interface is composed of two homogeneous materials, as shown in Figure 1. The two materials are isotropic, both having Poisson ratio of $\nu = 0.3$,

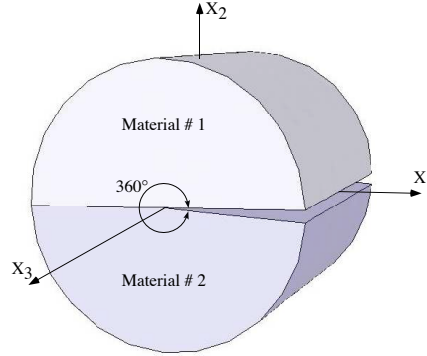


Fig. 1. *Bi-material cracked domain.*

with Young's modulus $E = 10$ of the upper material (material # 1) and $E = 1$ of the lower material (material # 2). This example was chosen to present a slightly simplified situation having an entire series expansion available. The first three eigen-values for this example problem are [5]:

$$\alpha_{1,2} = 0.5 \pm i0.07581177769 \quad , \quad \alpha_3 = 0.5 \quad (6)$$

The eigen-functions and first two shadow functions for this benchmark problem are given by (3) when using the constants C_k provided in Tables 1 and 2. The eigen-function and the first two shadow functions associated with the first two eigen-values are presented in Figure 2.

The eigen-pairs and shadows of example A are obtained numerically (see [3]) and therefore (8) represents an approximation of the exact functions. This approximation however is of very high accuracy of less than 0.1% relative error.

The chosen ESIFs are polynomials of degree 2:

$$A_{1,2}(x_3) = (3 + 4x_3 + 5x_3^2) \pm i(2 + 3x_3 + 4x_3^2), \quad A_3(x_3) = 5 + 4x_3 + 2x_3^2 \quad (7)$$

therefore the solution in the vicinity of the edge is:

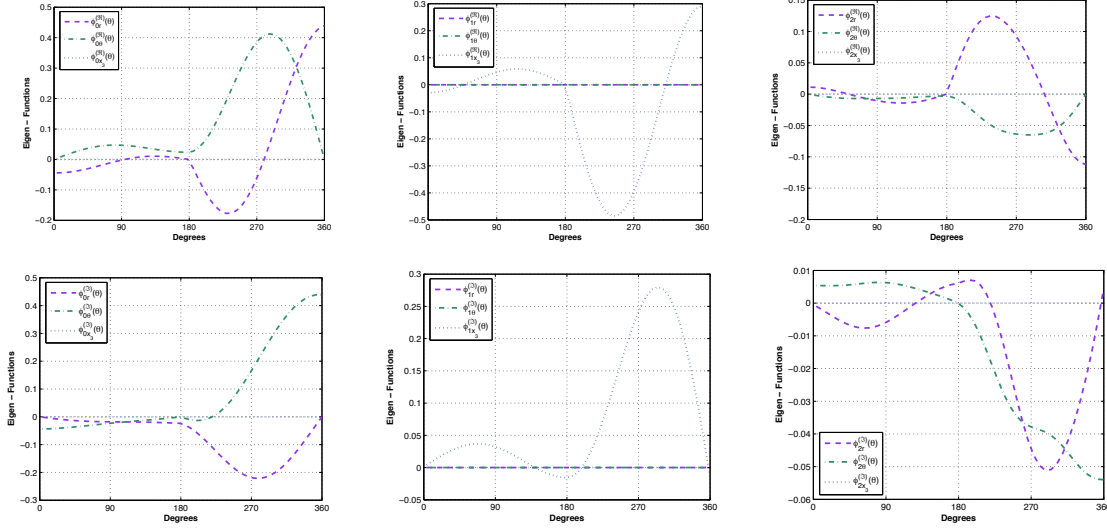


Fig. 2. The real and complex (top & bottom, respectively) part of the eigen-functions φ_0 , φ_1 , φ_2 associated with $\alpha_{1,2} = 0.5 \pm i0.075812$: example A.

$$\begin{aligned}
\mathbf{u} = & \left((3 + 4x_3 + 5x_3^2) + i(2 + 3x_3 + 4x_3^2) \right) r^{\alpha_{1\Re} + i\alpha_{1\Im}} \left(\varphi_{0\Re}^{(\alpha_1)}(\theta) + i\varphi_{0\Im}^{(\alpha_1)}(\theta) \right) (8) \\
& + (4 + 10x_3 + i(3 + 8x_3)) r^{1 + \alpha_{1\Re} + i\alpha_{1\Im}} \left(\varphi_{1\Re}^{(\alpha_1)}(\theta) + i\varphi_{1\Im}^{(\alpha_1)}(\theta) \right) \\
& + (10 + i8) r^{2 + \alpha_{1\Re} + i\alpha_{1\Im}} \left(\varphi_{2\Re}^{(\alpha_1)}(\theta) + i\varphi_{2\Im}^{(\alpha_1)}(\theta) \right) \\
& + \left((3 + 4x_3 + 5x_3^2) - i(2 + 3x_3 + 4x_3^2) \right) r^{\alpha_{1\Re} - i\alpha_{1\Im}} \left(\varphi_{0\Re}^{(\alpha_1)}(\theta) - i\varphi_{0\Im}^{(\alpha_1)}(\theta) \right) \\
& + (4 + 10x_3 - i(3 + 8x_3)) r^{1 + \alpha_{1\Re} - i\alpha_{1\Im}} \left(\varphi_{1\Re}^{(\alpha_1)}(\theta) - i\varphi_{1\Im}^{(\alpha_1)}(\theta) \right) \\
& + (10 - i8) r^{2 + \alpha_{1\Re} - i\alpha_{1\Im}} \left(\varphi_{2\Re}^{(\alpha_1)}(\theta) - i\varphi_{2\Im}^{(\alpha_1)}(\theta) \right) \\
& + (5 + 4x_3 + 2x_3^2) r^{\alpha_3} \varphi_0^{(\alpha_3)}(\theta) + 4(1 + x_3) r^{\alpha_3 + 1} \varphi_1^{(\alpha_3)}(\theta) + 4r^{\alpha_3 + 2} \varphi_2^{(\alpha_3)}(\theta)
\end{aligned}$$

For the domain in Figure 1 defined by $\{(r, \theta, x_3) | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq x_3 \leq 1\}$ on which traction free boundary conditions on the crack faces and Dirichlet boundary conditions on $r = 1$ and $x_3 = \pm 1$ are prescribed according to (7)-(8), one obtains the exact solution (7)-(8) at each r, θ, x_3 . The crack faces are traction free.

To examine the accuracy of series (8) we compute the traction at the two traction free crack surfaces, using (8) (having α_3 only) where $A_3 = 5 + 4x_3 + 2x_3^2$ and the C_k coefficients in Table 2. Although the traction should be zero (traction free boundary conditions on the two surfaces) the calculated traction is of an order of 10^{-5} due to small numerical errors.

Table 1

C_k of real part of the eigen-function, and first two shadow functions associated with $\alpha_{1,2} = 0.5 \pm i0.075812$ - example A.

	Real Part									
	$0 \leq \theta \leq 180$					$180 \leq \theta \leq 360$				
	φ_0		φ_1	φ_2		φ_0		φ_1	φ_2	
	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ
C_1	0	0	0	0	0	-0.008296	0.284280	-0.005842	-0.028608	-0.075810
C_2	0	0	0	0	0	0	0	0	0	0
C_3	0.001340	-0.044575	0.001865	-0.003996	0.008624	0	0	0	0	0
C_4	-0.000091	-0.000072	0.000430	-0.001656	0.000036	-0.003046	-0.000825	0.001219	0.016147	0.004843
C_5	0.001398	-0.021071	0.000732	-0.001924	0.005854	0.010789	-0.259810	0.004349	0.010160	0.068870
C_8	-0.022086	0.000175	0.020100	-0.005694	0.002366	-0.184330	-0.000335	0.128240	-0.036658	-0.030244
C_9	-0.021644	-0.000034	-0.050039	0.016421	-0.000913	-0.258360	0.002935	-0.415450	0.146510	0.011816
C_{11}	-0.000215	-0.000166	0.000803	0.000509	-0.001307	0.003889	-0.002810	-0.005295	0.003218	0.016573
	Imaginary Part									
	$0 \leq \theta \leq 180$					$180 \leq \theta \leq 360$				
	φ_0		φ_1	φ_2		φ_0		φ_1	φ_2	
	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ
C_1	0	0	0	0	0	-0.137680	0.019182	0.183620	-0.034211	-0.027624
C_2	0	0	0	0	0	0	0	0	0	0
C_3	0.027840	-0.002089	-0.013019	-0.001744	0.000338	0	0	0	0	0
C_4	-0.000381	-0.000461	-0.001204	-0.000522	-0.000182	0.004935	-0.003912	0.010488	0.002167	0.010478
C_5	0.004731	-0.001732	-0.027566	0.004913	0.000644	0.109710	-0.016178	-0.208420	0.035949	0.016541
C_8	0.003406	-0.040553	0.002676	-0.003072	0.007558	0.028426	-0.316950	0.027023	-0.006660	0.026137
C_9	-0.003448	-0.002680	-0.000873	0.001912	-0.001576	-0.028265	-0.129140	-0.008292	0.008246	0.019970
C_{11}	0.000039	-0.000763	-0.000399	0.000716	-0.000460	-0.000320	0.007610	-0.004628	-0.004693	0.007187

φ_{x_3} associated with φ_0 , $\varphi_r, \varphi_\theta$ associated with φ_1 and φ_{x_3} associated with φ_2 are zero. The constants $C_6, C_7, C_{10}, C_{12}, C_{13}, C_{14}$ are zero.

Table 2

C_k of eigen-functions and first two shadow functions associated with $\alpha_3 = 0.5$ - example A.

	$0 \leq \theta \leq 180$				$180 \leq \theta \leq 360$			
	φ_0	φ_1		φ_2	φ_0	φ_1		φ_2
	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_{x_3}	φ_r	φ_θ	φ_{x_3}
C_1	0	0	0	0	-0.000006	-0.000061	-0.240940	0.000042
C_2	0	0	0	0	0	0	0	0
C_3	0	-0.000005	0.024094	0.000001	0	0	0	0
C_4	0	-0.000002	0.024096	-0.000002	0	0.000018	0.240960	-0.000023
C_5	0	-0.000005	-0.000002	0.000002	0.000001	0.000047	-0.000017	-0.000017
C_8	0.072294	-0.024104	0.000004	-0.012046	0.722940	-0.241040	0.000073	-0.120460
C_9	0	0.000003	0.000003	-0.000002	0	0.000033	-0.000028	-0.000017
C_{11}	0	-0.024096	-0.000001	0.000802	0	-0.240960	0.000004	0.008026

$\varphi_r, \varphi_\theta$ associated with φ_0 , φ_{x_3} associated with φ_1 and φ_r, v associated with φ_2 are zero. The constants $C_6, C_7, C_{10}, C_{12}, C_{13}, C_{14}$ are zero.

3.2 Example G - CTS: Crack at the interface of two anisotropic materials

Consider the classical compact tension specimen (CTS) of a constant thickness $-1 < x_3 < 1$, shown in Figure 3. The CTS's faces are traction free and it is loaded by bearing loads at the tearing holes having an equivalent force of 100 in the x_2 direction as seen in Figure 4. Although the loading is independent of x_3 , because of the vertex singularities at $x_3 = \pm 1$ the ESIFs changes as the vertices are approached.

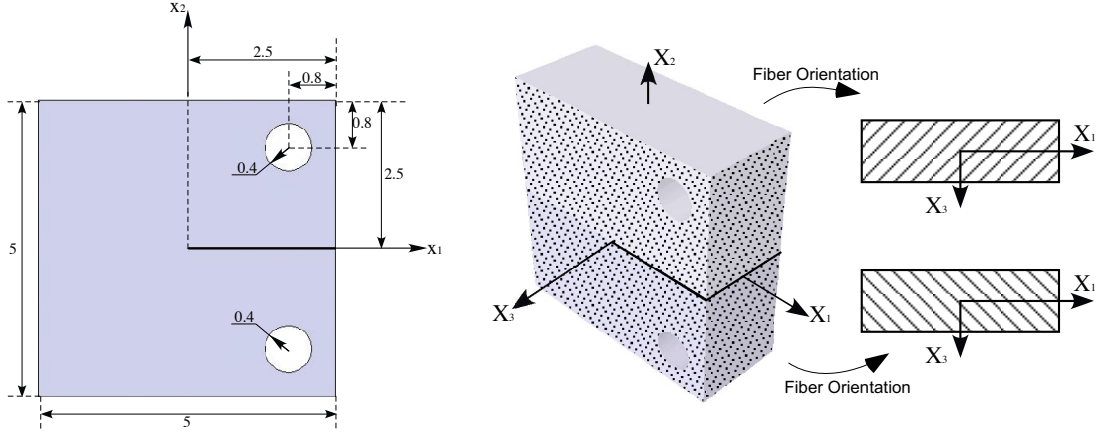


Fig. 3. Example G: Dimensions of the CTS (thickness is 2, $-1 < x_3 < 1$).

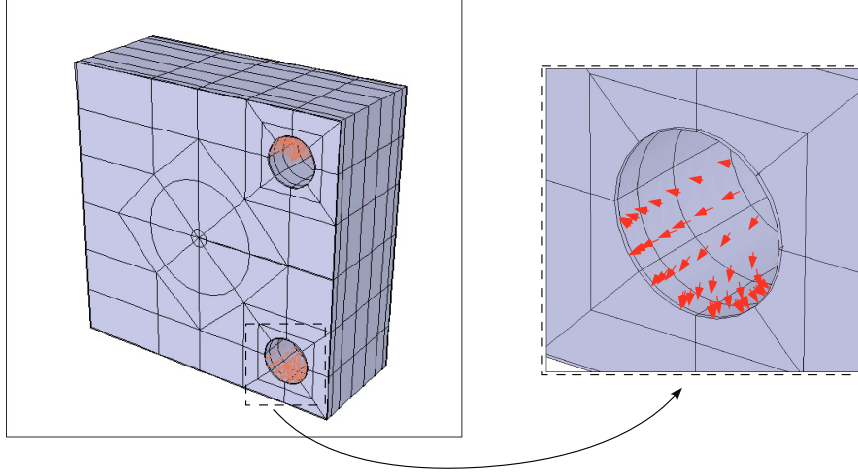


Fig. 4. Example G: The p -finite element mesh used for ESIF extraction.

The CTS is made of two orthotropic materials made of the same high-modulus graphite-epoxy composite with different fiber orientation. The material properties are:

$$\begin{aligned} E_L &= 1.38 \times 10^5 \text{MPa} & E_T &= E_z = 1.45 \times 10^4 \text{MPa} \\ G_{LT} &= G_{Lz} = G_{Tz} = 0.586 \times 10^4 \text{MPa} & \nu_{LT} &= \nu_{Lz} = \nu_{Tz} = 0.21 \end{aligned} \quad (9)$$

The subscripts L, T, z refer to fiber, transverse and thickness direction of the composite. The fibers orientation in the upper material is $+45^\circ$ and in the lower

material -45° .

The first three eigen-values for this example problem are:

$$\alpha_{1,2} = 0.5 \pm i0.034328 \quad , \quad \alpha_3 = 0.5 \quad (10)$$

Unlike Example A, herein an infinite number of shadow functions in (3) exist because the ESIFs are not necessary polynomials. The first three eigenfunctions of engineering importance (associated with eigen-values smaller than 1 resulting in singular stresses) and first two shadow functions (for each eigenfunction) in the solution (3) are given by the C_k 's provided in Tables 3 and 4. The first three ESIFs were computed by the quasi-dual function method in conjunction with the p-version of the finite element method (see FE mesh in Figure 4) according to [3] and presented in Figure 5.

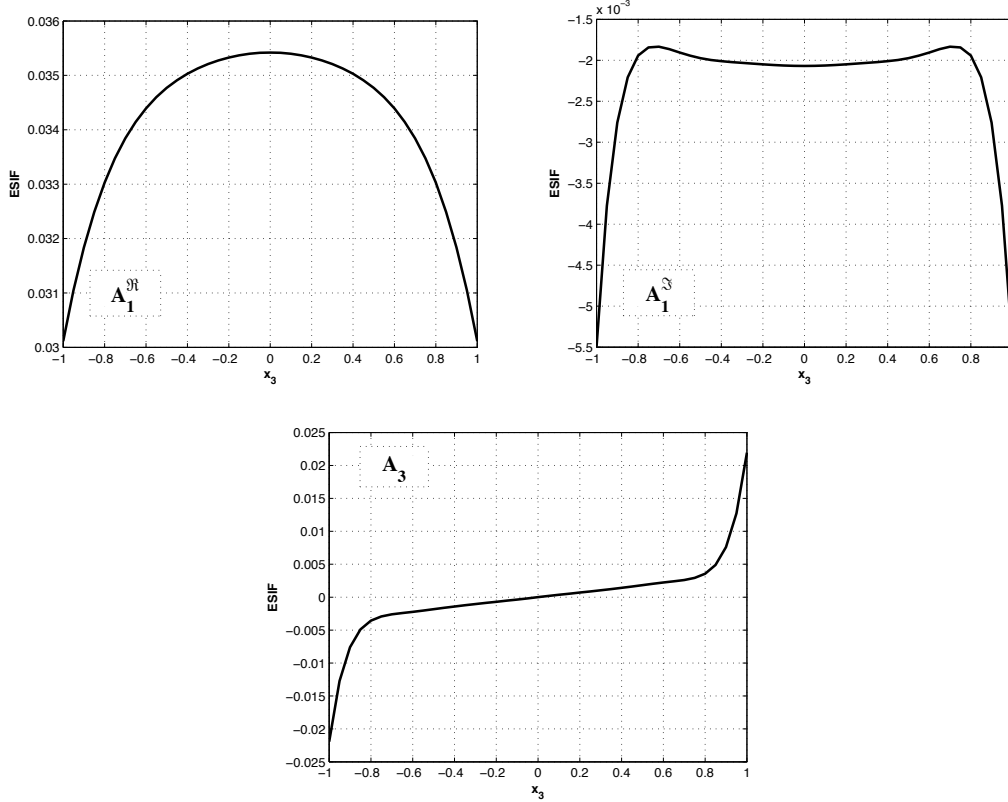


Fig. 5. ESIFs for example G: $A_1^R(x_3) \pm iA_1^S(x_3)$ and $A_3(x_3)$.

3.3 Example M - CTS: two cross-ply anisotropic materials

Consider a bi-material interface structure of constant thickness $-1 < x_3 < 1$, shown in Figure 6. This structure is similar to the CTS in previous section, having traction free boundary conditions on all faces except the bearing loads of a magnitude of 100

in the x_2 direction at the holes, see Figure 7. Because of the vertex singularities at $x_3 = \pm 1$ the first ESIF tends to infinity as the vertices are approached.

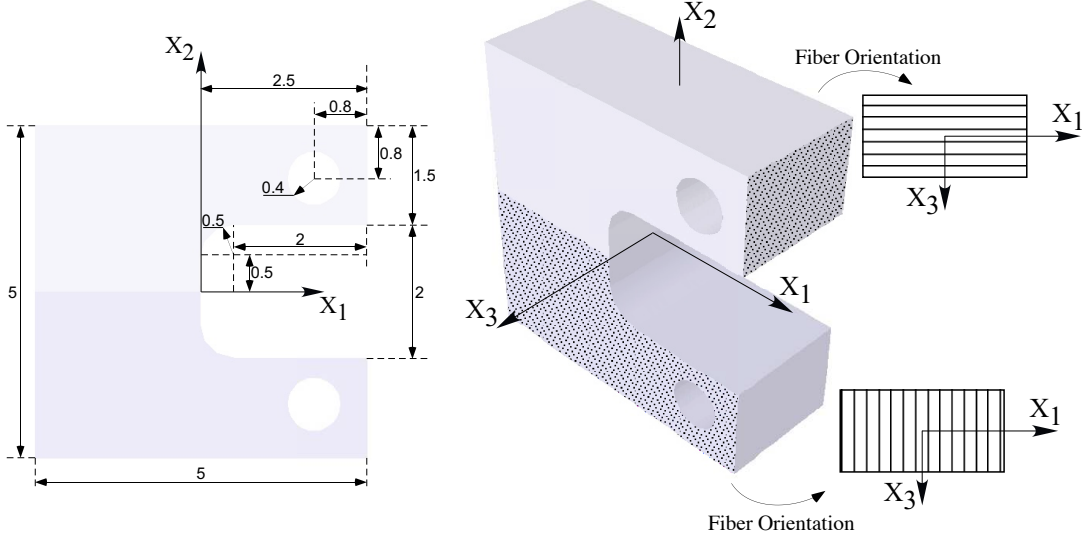


Fig. 6. Example M: Dimensions (thickness is 2, $-1 < x_3 < 1$).

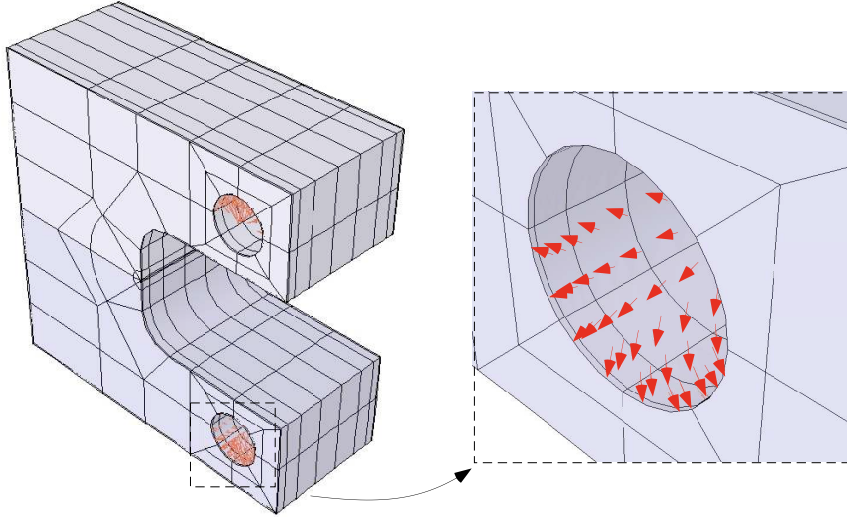


Fig. 7. Example M: The p -finite element mesh used for ESIF extraction.

This structure is made of two orthotropic materials made of the same high-modulus graphite-epoxy system with different fiber orientation. Material properties are given in (9) with fibers orientation in the upper material at x_1 direction whereas the orientation in the lower material is at x_3 direction.

Only the first eigen-value for this example problem results in a singular stress (and is of very mild singularity):

$$\alpha_1 = 0.966629 \quad (11)$$

other eigen-values are larger or equal to 1 and therefore are of no engineering importance. As for Example G, in this example there is an infinite number of shadow functions in (3) also. Only first eigenfunction and its shadows in the solution (3), are given by the C_k 's in Table 5.

The first ESIF (the only one of engineering relevance) is extracted and plotted in Figure 8. The influence of the vertex singularities at $x_3 = \pm 1$ is visible, resulting

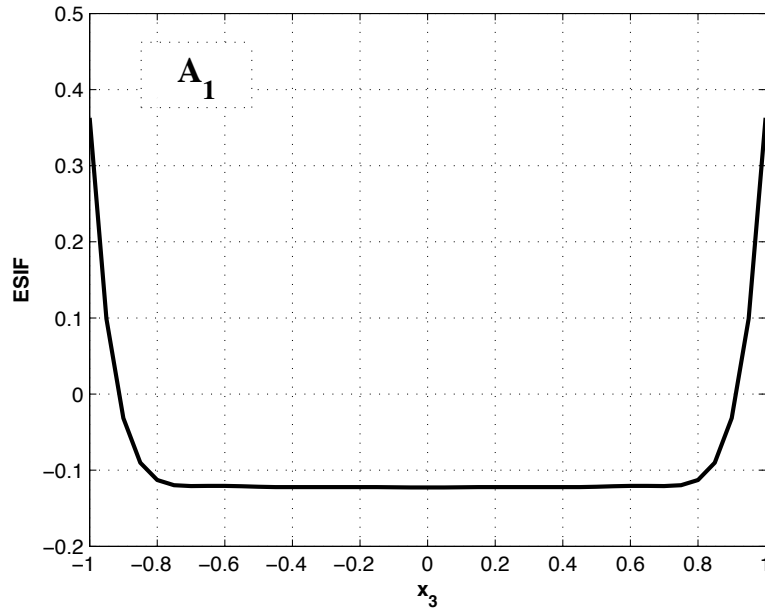


Fig. 8. *ESIF for the CTS M problem.*

in strong gradients in ESIF (which tends to ∞ at the vertices).

Table 3. C_k of real and imaginary parts of eigen-function and first two shadow functions associated with $\alpha_{1,2} = 0.5 \pm i0.034328$ - example G.

	Real part											
	$0 \leq \theta \leq 180$						$180 \leq \theta \leq 360$					
	φ_r	φ_θ	φ_{x3}	φ_r	φ_θ	φ_{x3}	φ_r	φ_θ	φ_{x3}	φ_r	φ_θ	φ_{x3}
C_1	1.348e-1	-2.101e-3	-1.537e-2	9.382e-2	5.842e-2	-1.871e-1	8.655e-2	-8.067e-2	-1.443e-1	1.348e-1	2.098e-3	1.537e-2
C_5	-5.574e-2	4.129e-3	1.088e-2	-4.735e-2	-3.793e-2	1.755e-1	-1.843e-2	1.035e-1	6.544e-2	-5.574e-2	-4.126e-3	-1.088e-2
C_6	6.193e-4	2.370e-3	9.856e-4	5.734e-3	-3.980e-3	-1.578e-3	1.060e-1	3.189e-2	-1.271e-3	6.190e-4	-2.370e-3	-9.857e-4
C_7	6.762e-3	-3.574e-4	-5.865e-3	4.158e-2	2.830e-2	-3.522e-2	3.592e-2	-9.591e-3	-7.835e-2	6.762e-3	3.571e-4	5.864e-3
C_8	5.452e-3	2.214e-1	2.508e-2	-9.701e-2	-3.337e-3	-1.337e-2	-1.045e-2	1.490e-2	2.108e-1	-5.450e-3	2.214e-1	2.508e-2
C_9	8.285e-4	5.948e-2	-1.598e-2	4.255e-2	-2.555e-2	3.310e-3	8.399e-2	-3.416e-1	-6.766e-2	-8.263e-4	5.948e-2	-1.598e-2
C_{11}	-2.399e-3	-4.623e-3	-1.014e-2	4.866e-2	-1.079e-2	8.030e-3	2.003e-3	-1.504e-2	-2.215e-1	2.399e-3	-4.623e-3	-1.014e-2
C_{13}	-3.913e-3	3.353e-5	1.189e-3	-2.576e-3	8.312e-3	6.760e-3	-3.875e-2	1.078e-1	3.570e-3	-3.913e-3	3.332e-5	1.189e-3
	Imaginary part											
	$0 \leq \theta \leq 180$						$180 \leq \theta \leq 360$					
	φ_r	φ_θ	φ_{x3}	φ_r	φ_θ	φ_{x3}	φ_r	φ_θ	φ_{x3}	φ_r	φ_θ	φ_{x3}
C_1	-1.169e-2	-3.952e-2	3.267e-2	-1.549e-1	-7.552e-1	-7.400e-3	6.075e-2	-1.522e-1	5.267e-2	-1.169e-2	3.952e-2	-3.267e-2
C_5	6.104e-3	5.206e-2	-1.698e-2	8.418e-2	1.324e-1	2.415e-2	-3.733e-2	1.599e-1	-3.313e-2	6.102e-3	-5.206e-2	1.698e-2
C_6	-3.088e-3	3.705e-4	-1.464e-3	8.604e-3	1.185e-2	1.699e-2	3.574e-2	8.235e-2	-8.551e-3	-3.088e-3	-3.704e-4	1.464e-3
C_7	-2.610e-3	1.238e-2	1.679e-2	-7.667e-2	-4.271e-1	2.701e-4	1.567e-2	-7.525e-2	2.449e-2	-2.611e-3	-1.238e-2	-1.679e-2
C_8	5.694e-2	2.019e-3	-3.062e-1	1.611e-1	-6.773e-2	3.256e-1	-1.188e-1	-7.371e-3	3.130e-1	-5.694e-2	2.019e-3	-3.062e-1
C_9	4.235e-2	1.573e-3	6.167e-2	-1.842e-2	3.561e-2	-3.269e-2	9.114e-2	-1.437e-1	-1.669e-1	-4.235e-2	1.575e-3	6.167e-2
C_{11}	1.586e-2	-1.977e-3	6.917e-3	-4.780e-1	3.487e-2	-8.955e-2	-2.767e-2	5.743e-3	1.331e-1	-1.586e-2	-1.976e-3	6.918e-3
C_{13}	8.287e-4	-1.629e-3	-2.902e-3	-6.410e-3	-2.833e-3	-6.556e-4	-8.647e-2	4.131e-2	1.205e-2	-8.289e-4	-1.629e-3	-2.902e-3

The constants $C_2, C_3, C_4, C_{10}, C_{12}, C_{14}$ are zero.

Table 4. C_k of eigen-function and first two shadow functions associated with $\alpha_3 = 0.5$ - example G.

	$0 \leq \theta \leq 180$												$180 \leq \theta \leq 360$											
	φ_0				φ_1				φ_2				φ_0				φ_1				φ_2			
	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}
C_1	-8.481e-3	2.835e-1	5.307e-2	-1.600e-2	5.635e-2	1.406e-2	8.483e-3	2.835e-1	5.307e-2	-4.022e-2	2.631e-2	-2.032e-2	8.483e-3	2.835e-1	5.307e-2	-1.600e-2	-5.635e-2	-1.406e-2	4.021e-2	2.631e-2	-2.032e-2	-8.481e-3	2.835e-1	5.307e-2
C_5	-2.947e-3	-1.511e-1	-2.749e-2	7.425e-3	-8.398e-2	-9.236e-3	2.941e-3	-1.511e-1	-2.749e-2	4.666e-2	4.702e-2	1.171e-2	2.941e-3	-1.511e-1	-2.749e-2	7.425e-3	8.398e-2	9.260e-3	-4.666e-2	4.702e-2	4.702e-2	1.171e-2	-2.947e-3	-1.511e-1
C_6	-7.449e-3	1.231e-4	-2.476e-3	5.865e-4	-2.151e-3	5.374e-4	7.448e-3	1.227e-4	-2.476e-3	1.550e-2	8.627e-2	1.169e-3	7.448e-3	1.227e-4	-2.476e-3	5.869e-4	2.151e-3	-5.373e-4	-1.549e-2	8.628e-2	8.628e-2	1.168e-3	-7.449e-3	1.231e-4
C_7	-4.162e-3	1.253e-2	2.747e-2	-9.211e-3	-2.545e-2	4.509e-3	4.163e-3	1.253e-2	2.747e-2	-9.553e-3	1.597e-2	-9.236e-3	4.163e-3	1.253e-2	2.747e-2	-9.209e-3	2.545e-2	-4.509e-3	9.553e-3	1.598e-2	1.598e-2	-9.241e-3	-4.162e-3	1.253e-2
C_8	-1.667e-1	-1.480e-3	7.436e-2	-7.950e-2	-3.087e-5	-1.286e-1	-1.667e-1	1.483e-3	-7.435e-2	6.227e-2	3.843e-2	-1.380e-1	-1.667e-1	1.483e-3	-7.435e-2	7.950e-2	-3.161e-5	-1.286e-1	6.227e-2	-3.843e-2	-3.843e-2	1.380e-1	-1.667e-1	
C_9	-1.588e-1	5.709e-3	6.206e-2	-8.422e-2	-3.830e-4	6.182e-1	-1.588e-1	-5.703e-3	-6.206e-2	-1.790e-1	-4.282e-2	8.877e-2	-1.588e-1	-5.703e-3	-6.206e-2	8.422e-2	-3.863e-4	6.182e-1	-1.790e-1	4.282e-2	4.282e-2	-8.877e-2	-1.588e-1	
C_{11}	1.997e-2	-2.793e-4	1.130e-2	-2.807e-2	3.932e-4	4.124e-3	1.997e-2	2.785e-4	-1.130e-2	1.505e-3	-9.447e-3	7.642e-2	1.997e-2	2.785e-4	-1.130e-2	2.807e-2	3.915e-4	4.125e-3	1.504e-3	9.445e-3	-7.642e-2	1.997e-2	-2.793e-4	
C_{13}	1.138e-3	-3.937e-3	-3.822e-3	4.263e-3	7.969e-5	-1.105e-3	1.138e-3	3.937e-3	3.822e-3	-8.213e-2	1.366e-2	-5.840e-3	1.138e-3	3.937e-3	3.822e-3	-4.263e-3	7.989e-5	-1.105e-3	-8.213e-2	-1.365e-2	-1.365e-2	5.840e-3	1.138e-3	

The constants $C_2, C_3, C_4, C_{10}, C_{12}, C_{14}$ are zero.

Table 5. C_k of first eigen-function and first two shadow functions associated with $\alpha_1 = 0.966629$: example M.

	$0 \leq \theta \leq 180$						$180 \leq \theta \leq 360$					
	φ_0		φ_1		φ_2		φ_0		φ_1		φ_2	
	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}	φ_r	φ_θ	φ_{x_3}
C_1	-0.057525	0.009735	0.006953	-2.467100	-0.328100	0.002383	0.002383	-0.000050	-0.000347	1.810700	-0.638130	-0.000347
C_2	0.008685	0.016553	-0.167350	0.075265	0.240510	0.009705	0.009705	0.058856	-0.168960	-0.006105	-0.216040	-0.168960
C_3	-1.434800	0.705620	0.244180	-67.207000	5.058700	-0.194930	-0.194930	-0.147380	0.113370	-56.949000	4.699900	0.113370
C_8	0.039866	-0.035057	-0.003761	2.227400	-0.720650	0.002422	0.002422	0.000958	-0.000354	2.168900	0.243620	-0.000354
C_9	-0.023477	0.010291	-0.026977	-0.520830	0.101050	-0.058518	-0.058518	0.009767	0.009697	0.449220	-0.004979	0.009697
C_{10}	-0.031351	0.355330	0.007513	-1.706100	0.619140	-0.059358	-0.059358	0.315730	0.044530	-4.468700	0.777580	0.044530

φ_{x_3} associated with φ_0 , $\varphi_r, \varphi_\theta$ associated with φ_1 and φ_{x_3} associated with φ_2 are zero. The constants $C_4 - C_7$ and $C_{11} - C_{14}$ are zero.

4 Summary

The elastic solution in the vicinity of edges in anisotropic multi-material interfaces representative of composite materials was provided for three different problems. These solutions, although being computed by numerical methods presented in [3] are of very high accuracy, therefore suggested as benchmarks for the verification of various methods aimed at computing edge stress intensity functions in composite-like structures.

References

- [1] M. Costabel, M. Dauge, and Z. Yosibash. A quasidual function method for extracting edge stress intensity functions. *SIAM Jour. Math. Anal.*, 35(5):1177–1202, 2004.
- [2] Y. Freed and L. Banks-Sills. A through interface crack between a $\pm 45^\circ$ transversely isotropic pair of materials. *Int. Jour. Fracture*, (133):1–41, 2005.
- [3] N. Omer and Z. Yosibash. Edge singularities in 3-d elastic anisotropic and multi-material domains. *Computer Meth. Appl. Mech. Engrg.*, 2007. Submitted for publication.
- [4] Z. Suo. Singularities, interfaces and cracks in dissimilar anisotropic media. *Proc. Royal Soc. London*, A427:331 – 358, 1990.
- [5] Z. Yosibash. Computing edge singularities in elastic anisotropic three-dimensional domains. *Int. Jour. Fracture*, 86(3):221–245, 1997.
- [6] Z. Yosibash. On solutions of two-dimensional linear elastostatic and heat-transfer problems in the vicinity of singular points. *Int. Jour. Solids and Structures*, 34(2):243–274, 1997.
- [7] Z. Yosibash and N. Omer. Numerical methods for extracging edge stress intensity functions in anisotropic three-dimensional domains. *Computer Meth. Appl. Mech. Engrg.*, 2007. Online version at - doi:10.1016j.cma.2006.10.017.
- [8] Z. Yosibash, N. Omer, M. Costabel, and M. Dauge. Edge stress intensity functions in polyhedral domains and their extraction by a quasidual function method. *Int. Jour. Fracture*, 136:37 – 73, 2005.